Multivariate cryptography

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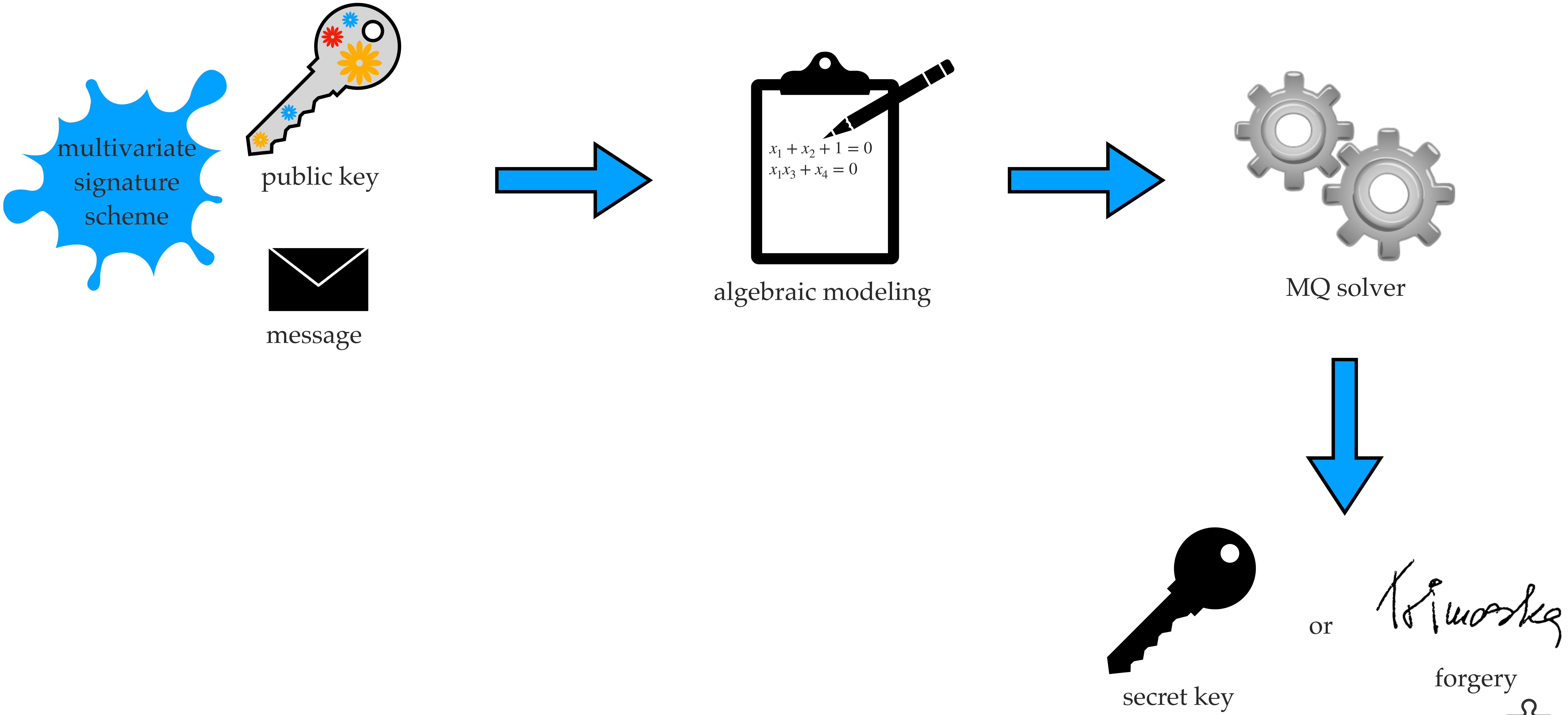
TU/e

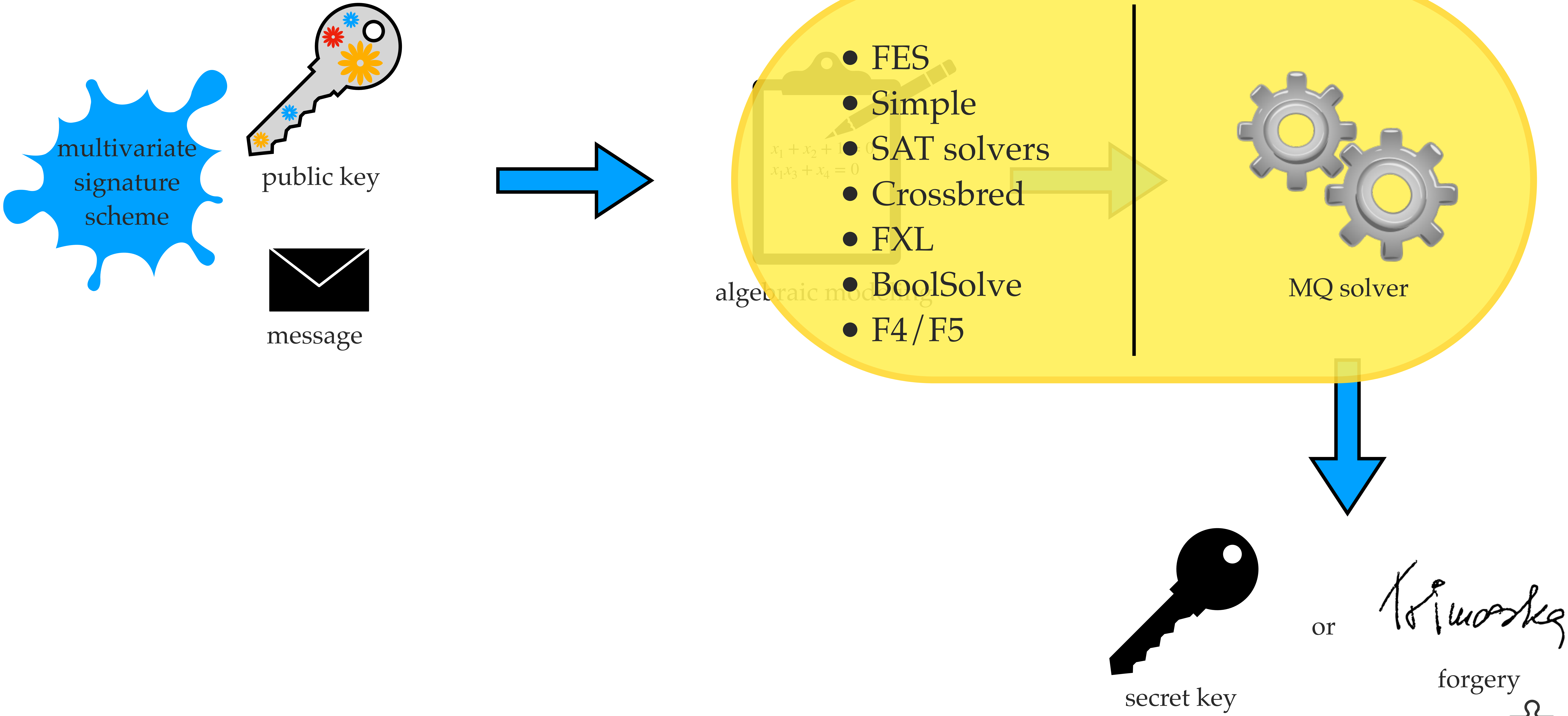
Algebraic cryptanalysis

A type of cryptanalytic methods where the problem of finding the secret key (or any attack goal) is reduced to the problem of finding a solution to a nonlinear multivariate polynomial system of equations.

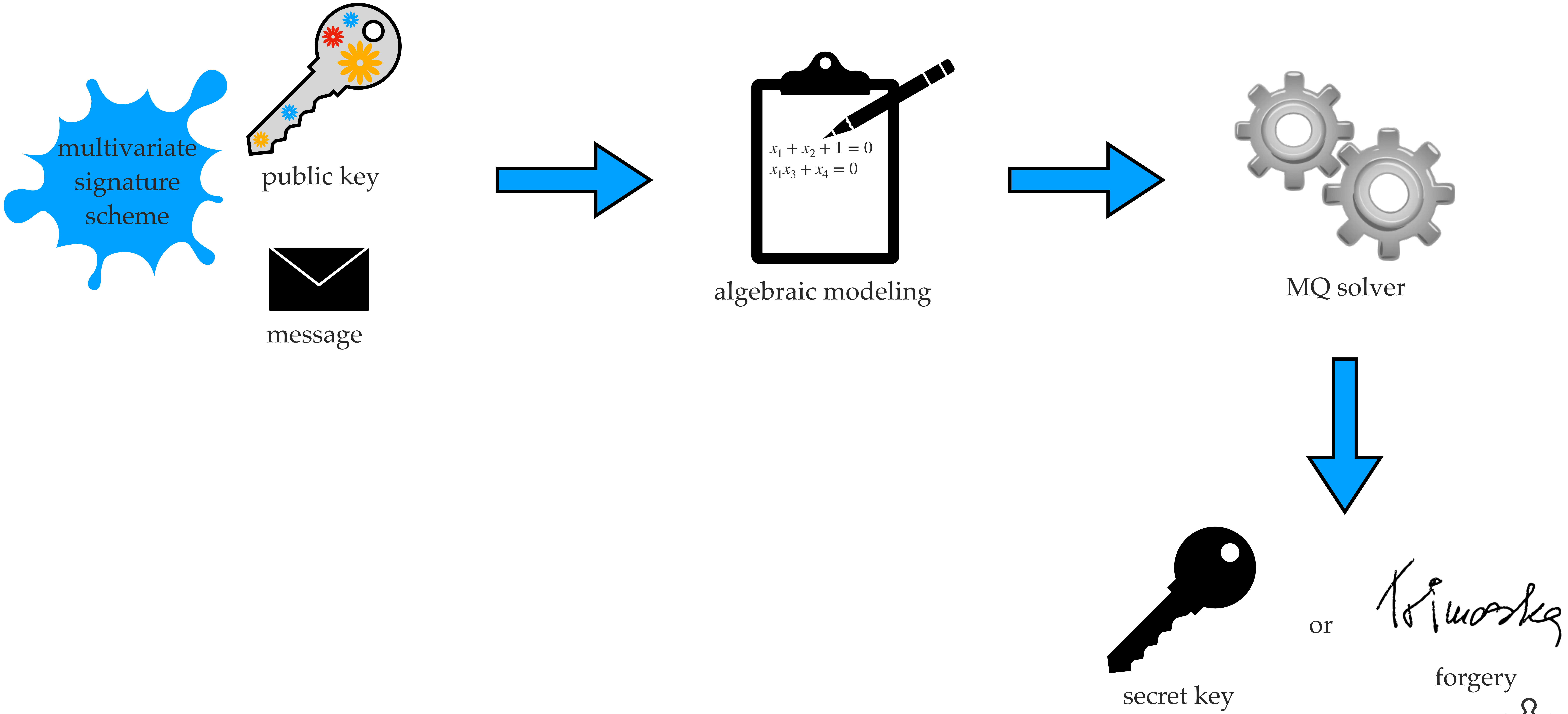
Algebraic cryptanalysis



Algebraic cryptanalysis



Algebraic cryptanalysis



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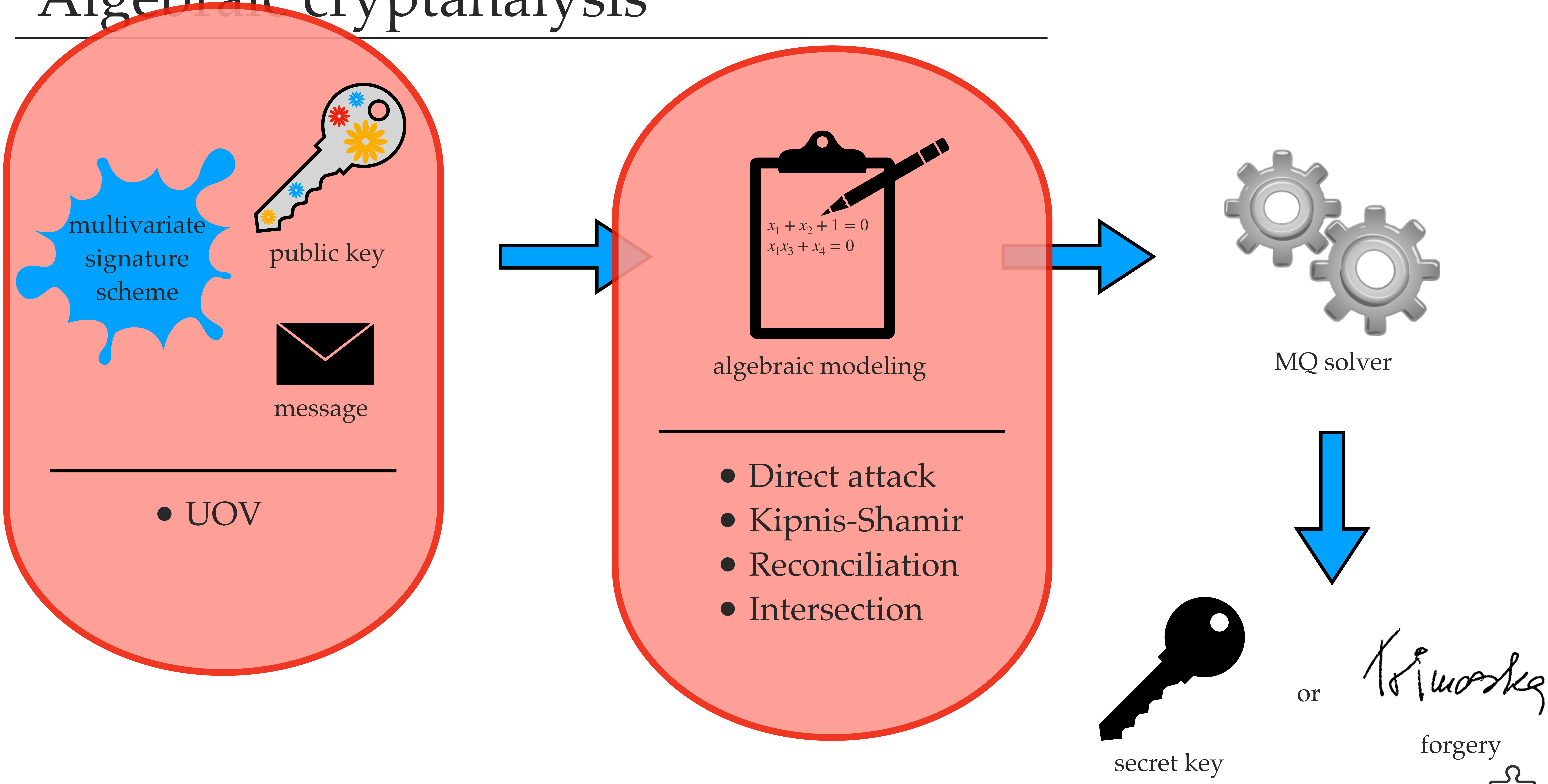


The MQ problem

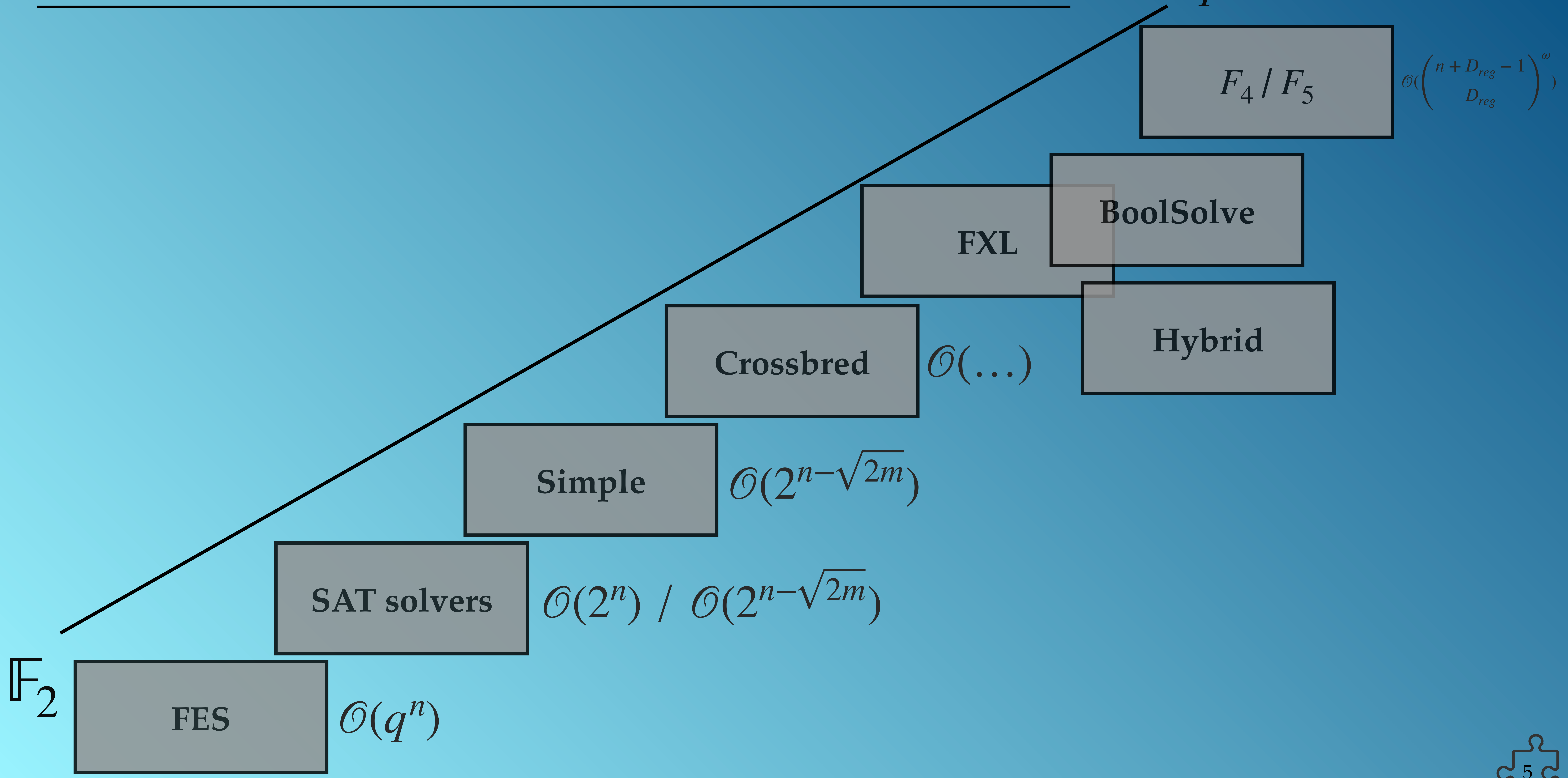
The MQ problem

Given m multivariate quadratic polynomials $f_1, \ldots, f_m$ of n variables over a finite field $\mathbb{F}_q$, find a tuple $\mathbf{x} = (x_1, \ldots, x_n)$ in $\mathbb{F}_q^n$, such that $f_1(\mathbf{x}) = \ldots = f_m(\mathbf{x}) = 0$.

Example.

$$f_1 : x_1x_3 + x_2x_4 + x_1 + x_3 + x_4 = 0$$
$$f_2 : x_2x_3 + x_1x_4 + x_3x_4 + x_1 + x_2 + x_4 = 0$$
$$f_3 : x_2x_4 + x_3x_4 + x_1 + x_3 + 1 = 0$$
$$f_4 : x_1x_2 + x_1x_3 + x_2x_3 + x_3 + x_4 + 1 = 0$$
$$f_5 : x_1x_2 + x_2x_3 + x_1x_4 + x_3 = 0$$
$$f_6 : x_1x_3 + x_1x_4 + x_3x_4 + x_1 + x_2 + x_3 + x_4 = 0$$

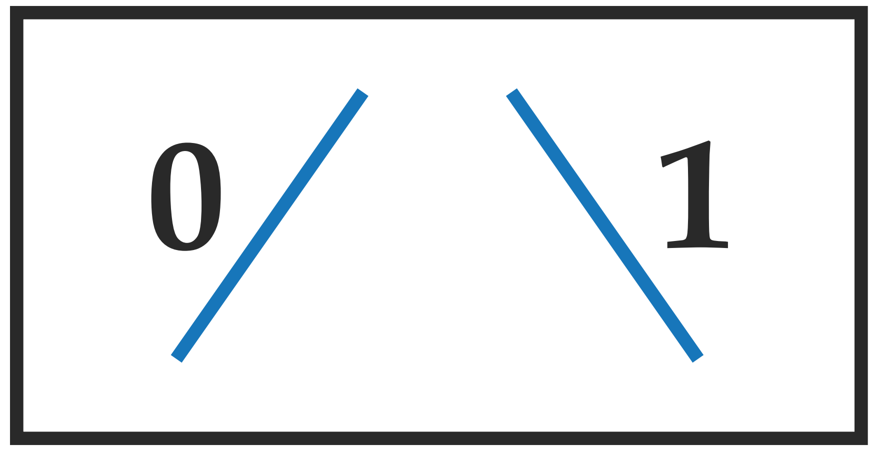
Overview of solvers

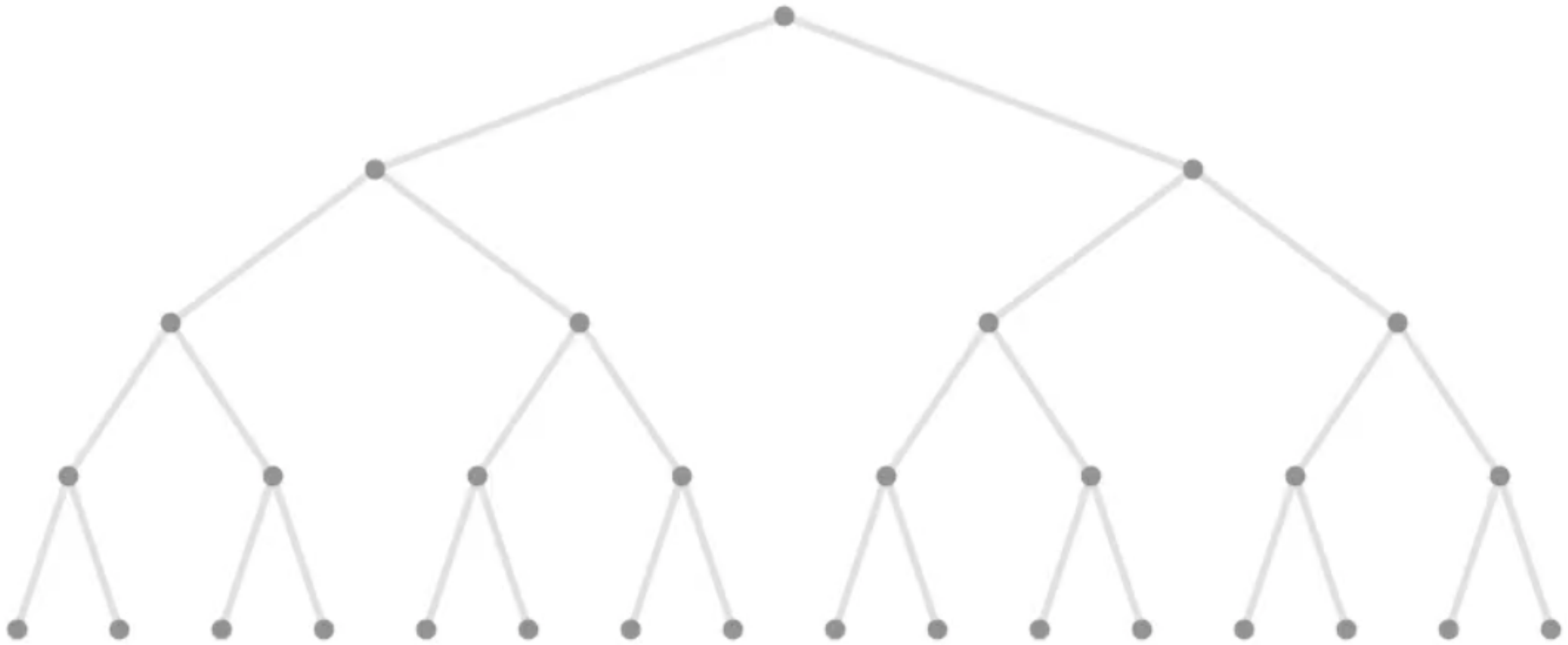


(Fast) Exhaustive Search

[Bouillaguet, Chen, Cheng, Chou, Niederhagen, Shamir, Yang, 2010]

Exhaustive Search





Binary search tree

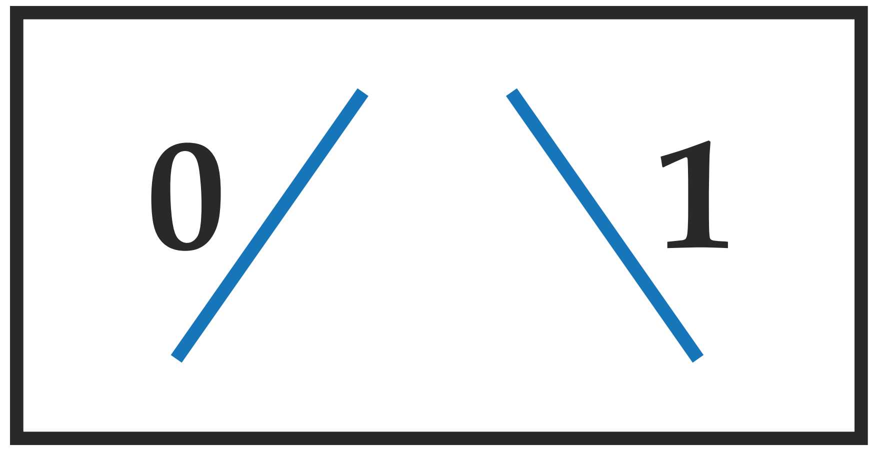
$$x_1 \cdot x_2 + x_1 \cdot x_3 + x_3 \cdot x_4 + x_3 = 0$$

$$x_2 \cdot x_3 + x_2 \cdot x_4 + x_1 + x_2 + 1 = 0$$

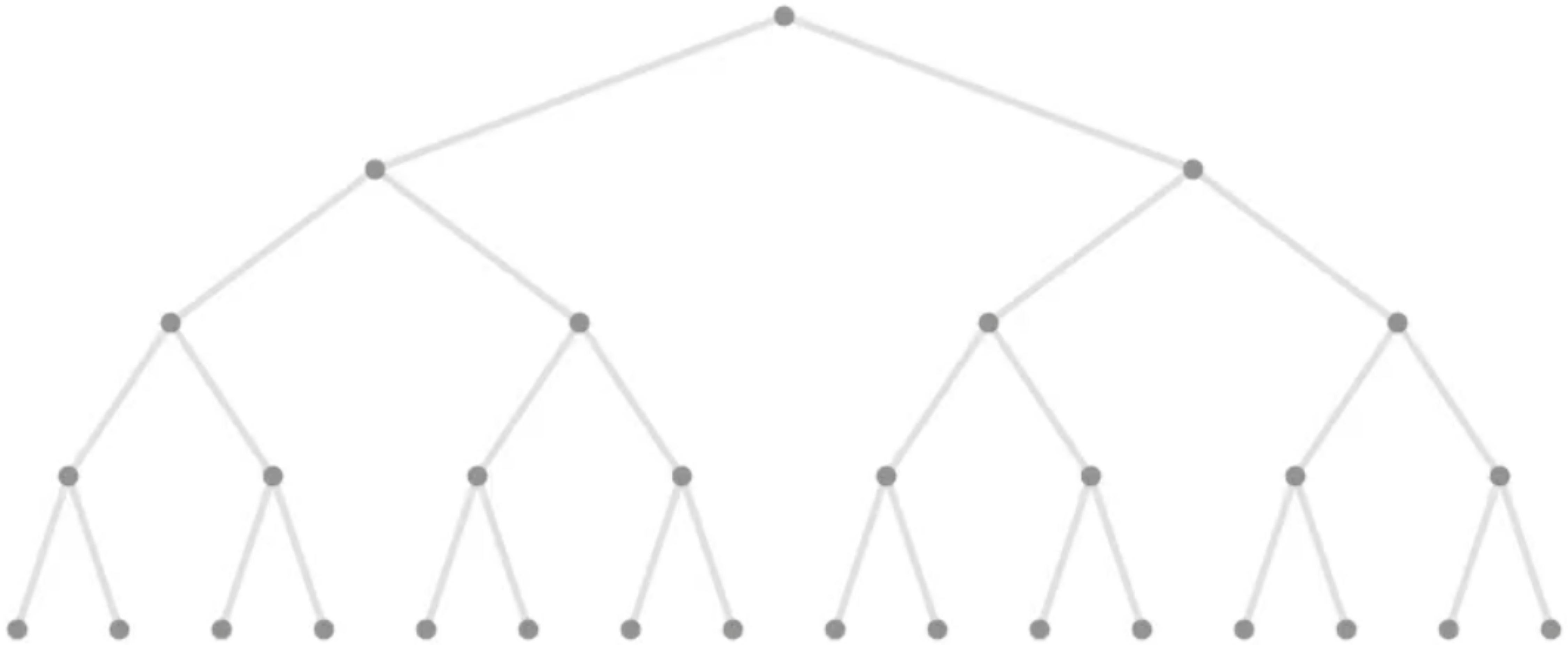
$$x_1 \cdot x_2 + x_2 \cdot x_3 + x_2 \cdot x_4 + x_1 + x_4 = 0$$

$$x_1 \cdot x_4 + x_2 \cdot x_3 + x_2 + x_3 + x_4 = 0$$

Exhaustive Search



Worst-case complexity: $\mathcal{O}(2^n)$



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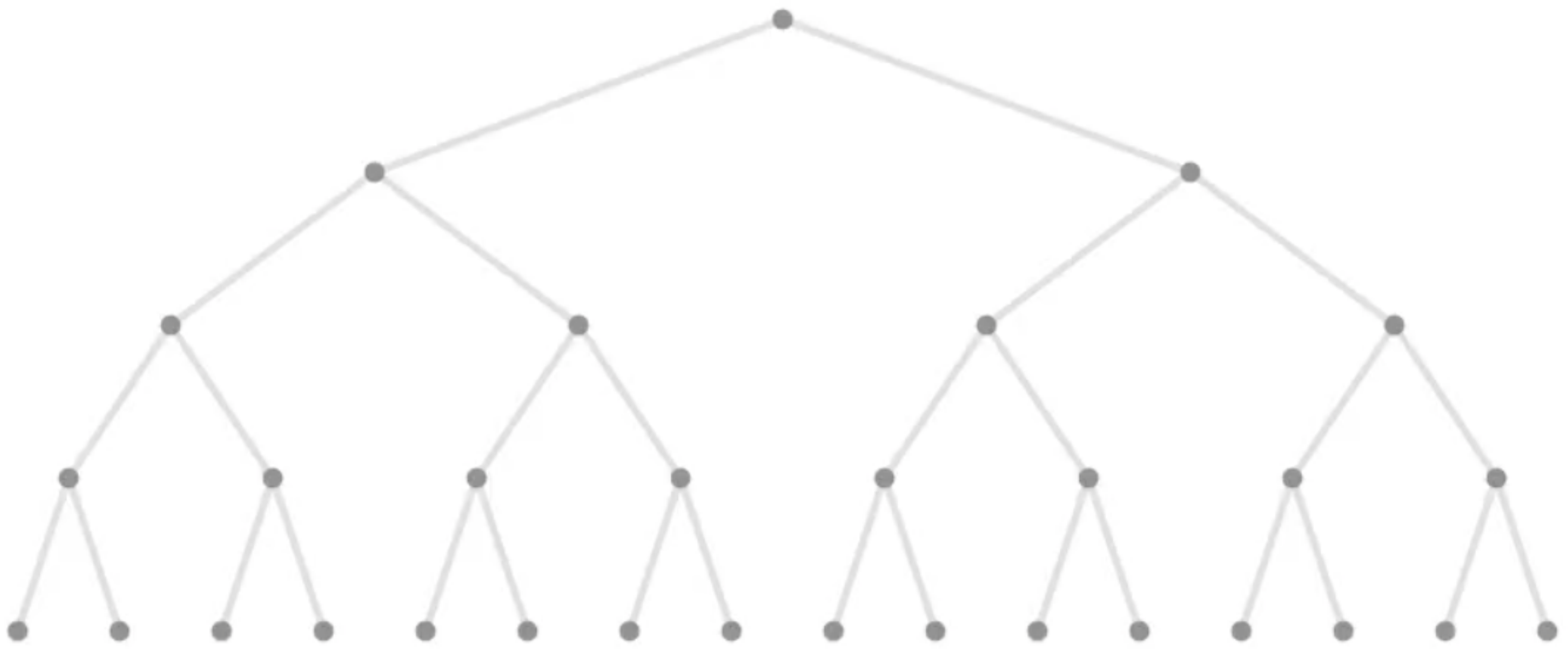

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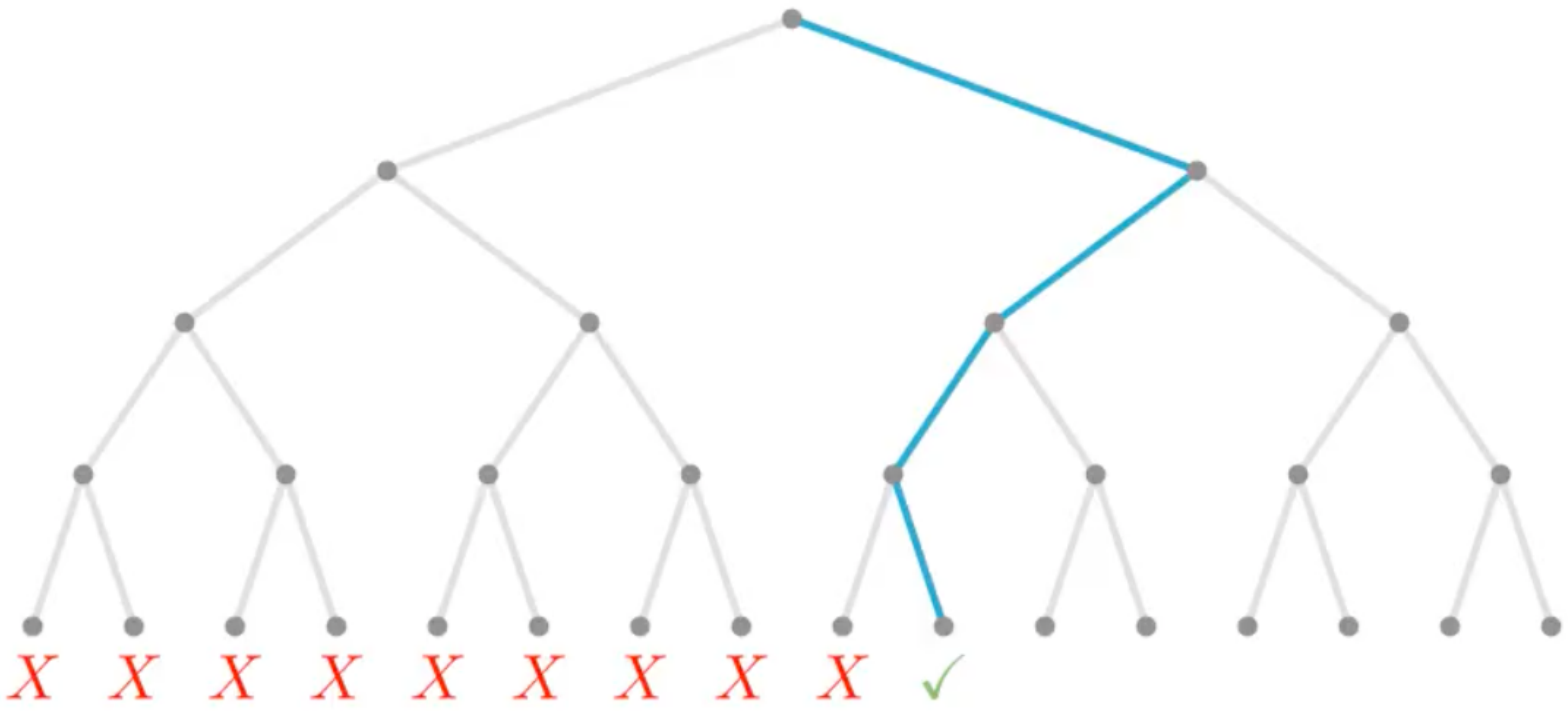

Binary search tree

$$1 \cdot 0 + 1 \cdot 0 + 0 \cdot 1 + 0 = 0$$

$$0 \cdot 0 + 0 \cdot 1 + 1 + 0 + 1 = 0$$

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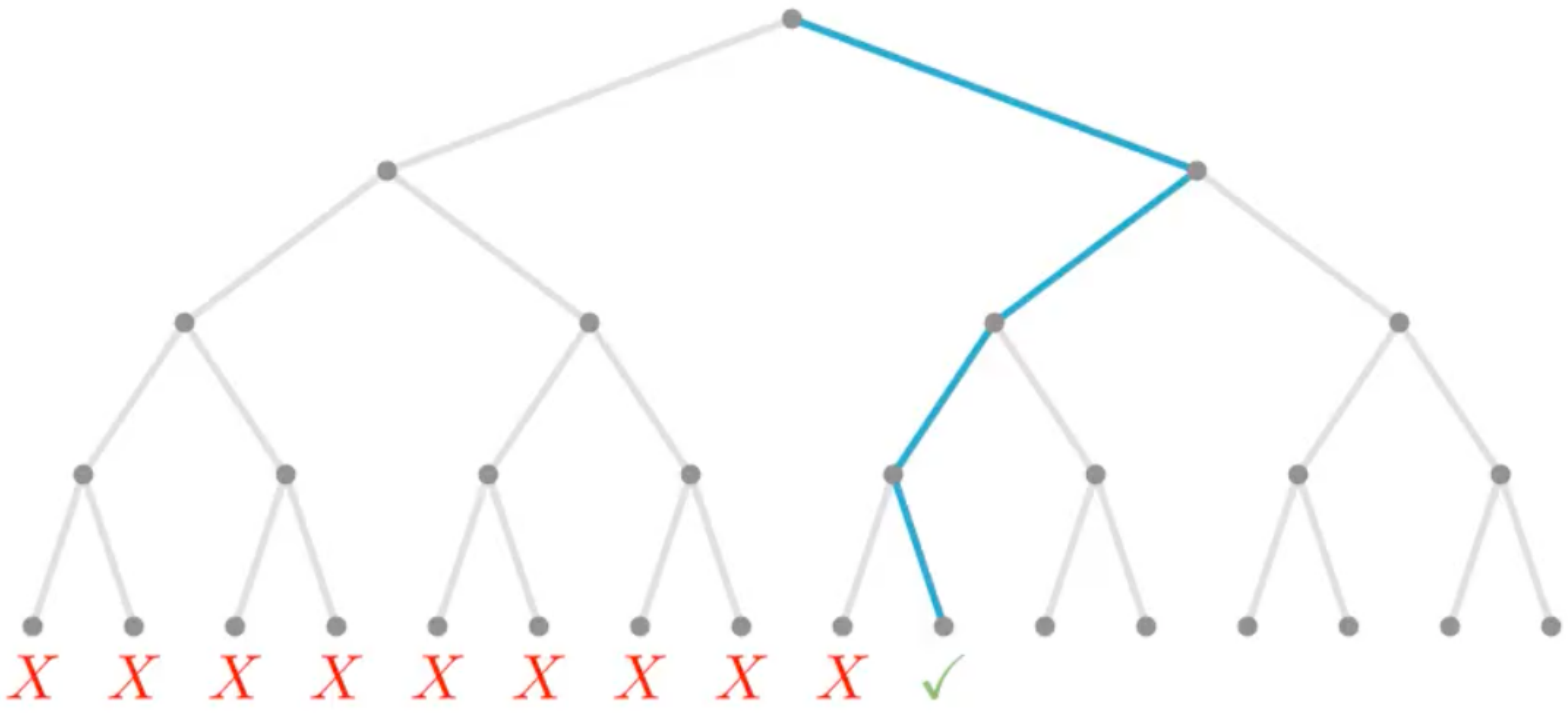

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Fast Exhaustive Search

* The libFES solver

Gray code

- An ordering of the binary system where two successive values differ in only one bit.

Example. $n = 4$

0000	1100
0001	1101
0011	1111
0010	1110
0110	1010
0111	1011
0101	1001
0100	1000

Fast Exhaustive Search

Gray code

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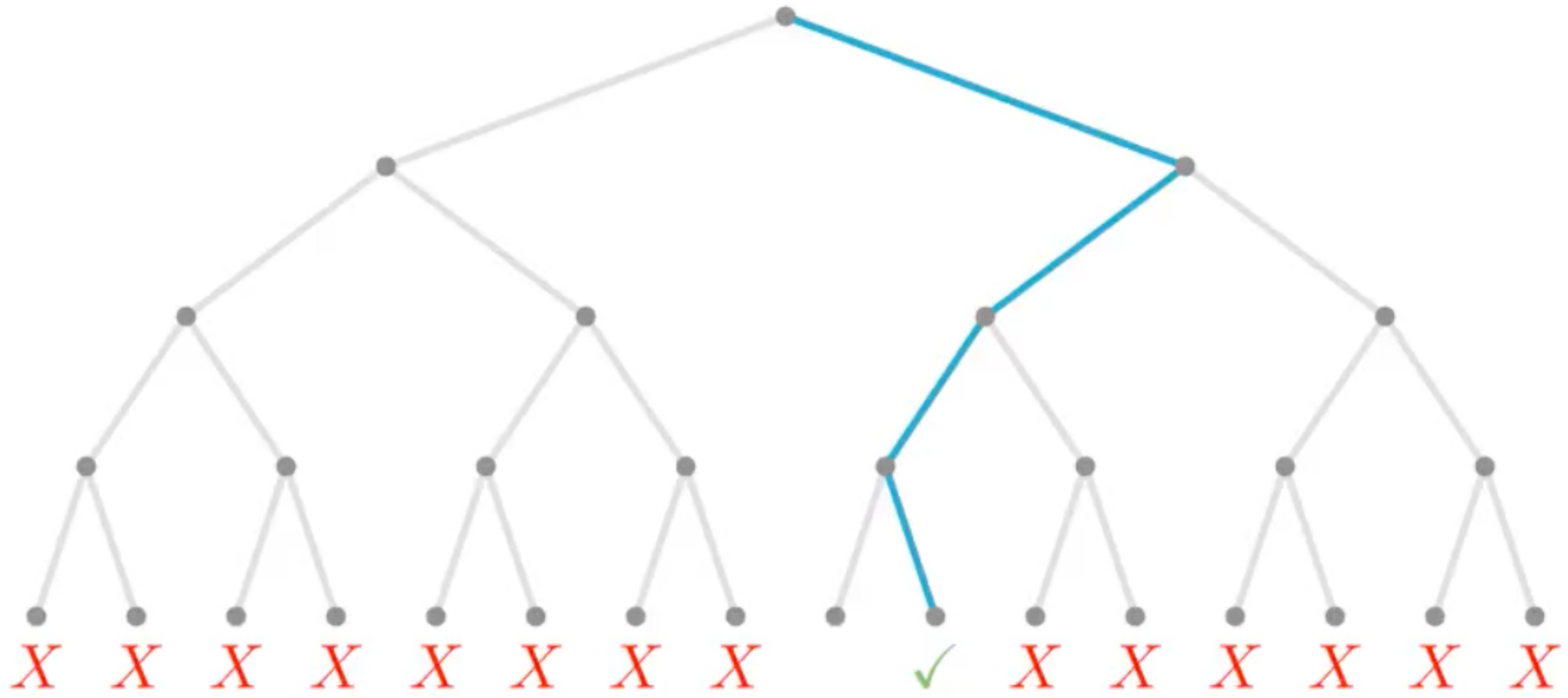

$$1 \cdot 0 + 1 \cdot 0 + 0 \cdot 1 + 0 = 0$$

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$$1 \cdot 1 + 0 \cdot 0 + 0 + 0 + 1 = 0$$

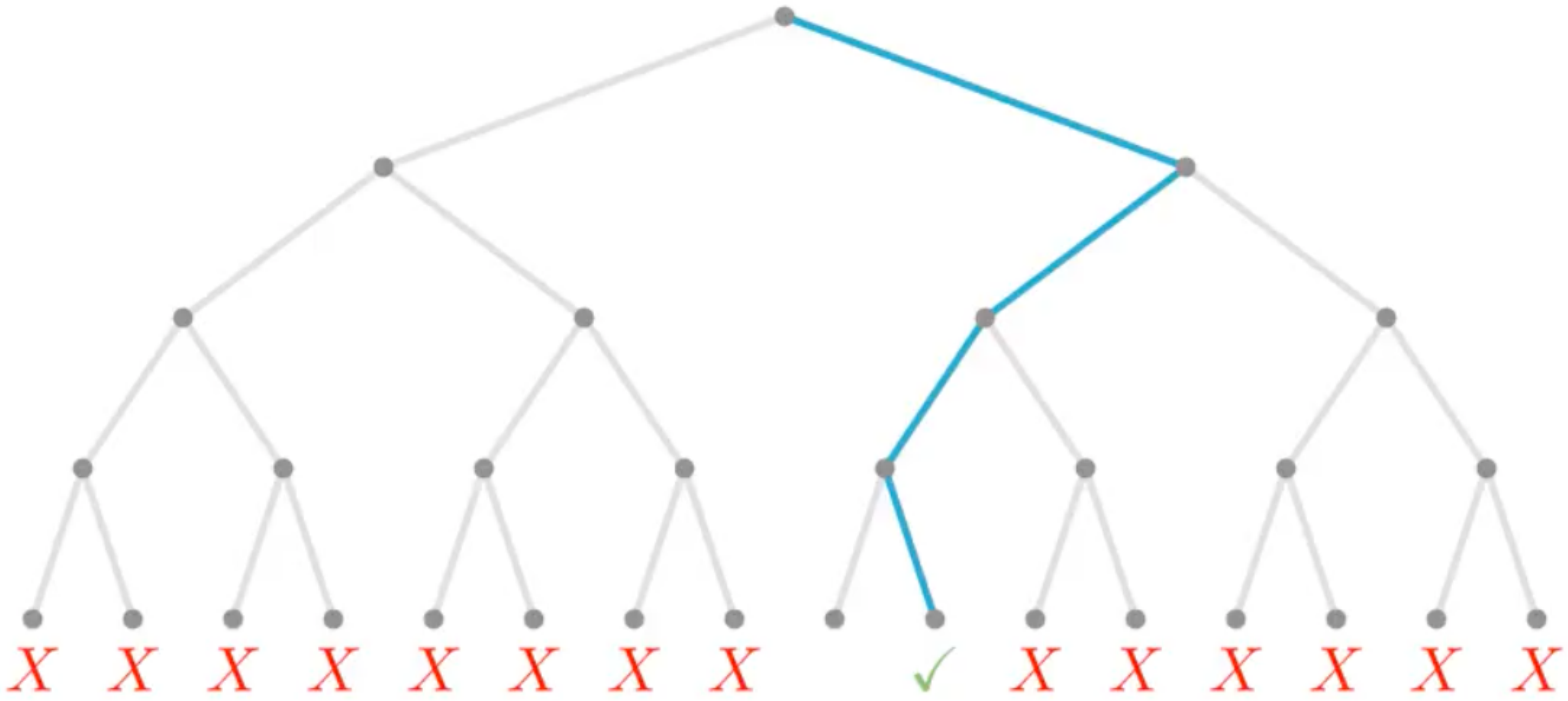
Fast Exhaustive Search

Worst-case complexity: $\mathcal{O}(2^n)$

! But, it differs from the depth-first traversal in the polynomial factors

Gray code

0000	1100
0001	1101
0011	1111
0010	1110
0110	1010
0111	1011
0101	1001
0100	1000



$$1 \cdot 0 + 1 \cdot 0 + 0 \cdot 1 + 0 = 0$$

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Macaulay matrix

Linearisation

Linear systems are easy to solve, nonlinear systems are hard.

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→ Linearisation: for each nonlinear monomial, replace all of its occurrences by a new variable.

Example.

$$
\begin{aligned}
f_1 &: x_1x_3 + x_2x_4 + x_1 + x_3 + x_4 = 0\\
f_2 &: x_2x_3 + x_1x_4 + x_3x_4 + x_1 + x_2 + x_4 = 0\\
f_3 &: x_2x_4 + x_3x_4 + x_1 + x_3 + 1 = 0\\
f_4 &: x_1x_2 + x_1x_3 + x_2x_3 + x_3 + x_4 + 1 = 0\\
f_5 &: x_1x_2 + x_2x_3 + x_1x_4 + x_3 = 0\\
f_6 &: x_1x_3 + x_1x_4 + x_3x_4 + x_1 + x_2 + x_3 + x_4 = 0
\end{aligned}
$$

→

$$
\begin{aligned}
f_1 &: y_2 + y_5 + x_1 + x_3 + x_4 = 0\\
f_2 &: y_4 + y_3 + y_6 + x_1 + x_2 + x_4 = 0\\
f_3 &: y_5 + y_6 + x_1 + x_3 + 1 = 0\\
f_4 &: y_1 + y_2 + y_4 + x_3 + x_4 + 1 = 0\\
f_5 &: y_1 + y_4 + y_3 + x_3 = 0\\
f_6 &: y_2 + y_3 + y_6 + x_1 + x_2 + x_3 + x_4 = 0
\end{aligned}
$$

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Linear systems are easy to solve, nonlinear systems are hard.

Linearisation: for each nonlinear monomial, replace all of its occurrences by a new variable.

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f_5 &: x_1x_2 + x_2x_3 + x_1x_4 + x_3 = 0\\
f_6 &: x_1x_3 + x_1x_4 + x_3x_4 + x_1 + x_2 + x_3 + x_4 = 0
\end{aligned}
\quad\longrightarrow\quad
\begin{aligned}
f_1 &: y_2 + y_5 + x_1 + x_3 + x_4 = 0\\
f_2 &: y_4 + y_3 + y_6 + x_1 + x_2 + x_4 = 0\\
f_3 &: y_5 + y_6 + x_1 + x_3 + 1 = 0\\
f_4 &: y_1 + y_2 + y_4 + x_3 + x_4 + 1 = 0\\
f_5 &: y_1 + y_4 + y_3 + x_3 = 0\\
f_6 &: y_2 + y_3 + y_6 + x_1 + x_2 + x_3 + x_4 = 0
\end{aligned}
$$

Linearisation

Linearisation adds solutions: a *random* quadratic system of m equations in n variables, when $n = m$, is expected to have one solution (probability is $\sim \frac{1}{q}$ for systems over $\mathbb{F}_q$). The corresponding linearised system has a solution space of dimension $\binom{n+1}{2} - m$.

$\binom{n}{2}$ quadratic plus n linear monomials

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$\binom{n}{2}$ quadratic plus n linear monomials

Loss of information: e.g. assignment $x_1 = 1; x_2 = 0; y_1 = 1;$ is part of a valid solution to the linearised system, but $x_1 x_2 \neq y_1$.

Macaulay matrix

Monomials →

Equations ↓

	x_1x_2	x_1x_3	x_1x_4	x_1	x_2x_3	x_2x_4	x_2	x_3x_4	x_3	x_4	1
f_1											
f_2											
f_3											
f_4											
f_5											
f_6											

$$f_1 : x_1x_3 + x_2x_4 + x_1 + x_3 + x_4 = 0$$
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	x_1x_2	x_1x_3	x_1x_4	x_1	x_2x_3	x_2x_4	x_2	x_3x_4	x_3	x_4	1
f_1	0	1	0	1	0	1	0	0	1	1	0
f_2	0	0	1	1	1	0	1	1	0	1	0
f_3	0	0	0	1	0	1	0	1	1	0	1
f_4	1	1	0	1	1	0	0	0	1	1	1
f_5	1	0	1	1	1	0	0	0	1	0	0
f_6	0	1	1	1	0	0	1	1	1	1	0

$f_1 : x_1x_3 + x_2x_4 + x_1 + x_3 + x_4 = 0$

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$f_3 : x_2x_4 + x_3x_4 + x_1 + x_3 + 1 = 0$

$f_4 : x_1x_2 + x_1x_3 + x_2x_3 + x_3 + x_4 + 1 = 0$

$f_5 : x_1x_2 + x_2x_3 + x_1x_4 + x_3 = 0$

$f_6 : x_1x_3 + x_1x_4 + x_3x_4 + x_1 + x_2 + x_3 + x_4 = 0$

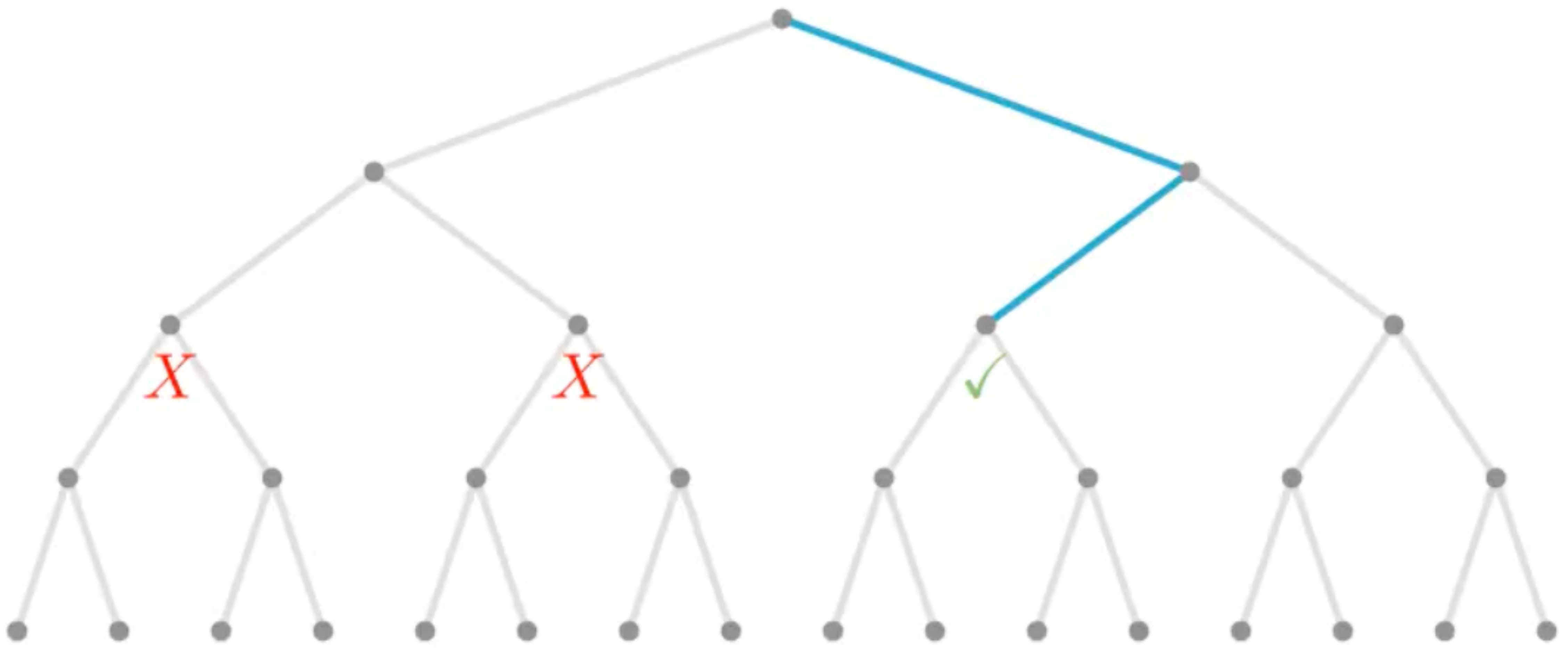
SAT solvers

CryptoMiniSat [Soos, Nohl, Castelluccia, 2009], WDSat [T., Dequen, Ionica, 2020]

Simple algorithm

[Bouillaguet, Delaplace, T., 2021]

Partial assignment and conflicts



$$1 \cdot 0 + 1 \cdot x_3 + x_3 \cdot x_4 + x_3 = 0$$

$$0 \cdot x_3 + 0 \cdot x_4 + 1 + 0 + 1 = 0$$

$$1 \cdot 0 + 0 \cdot x_3 + 0 \cdot x_4 + 1 + x_4 = 0$$

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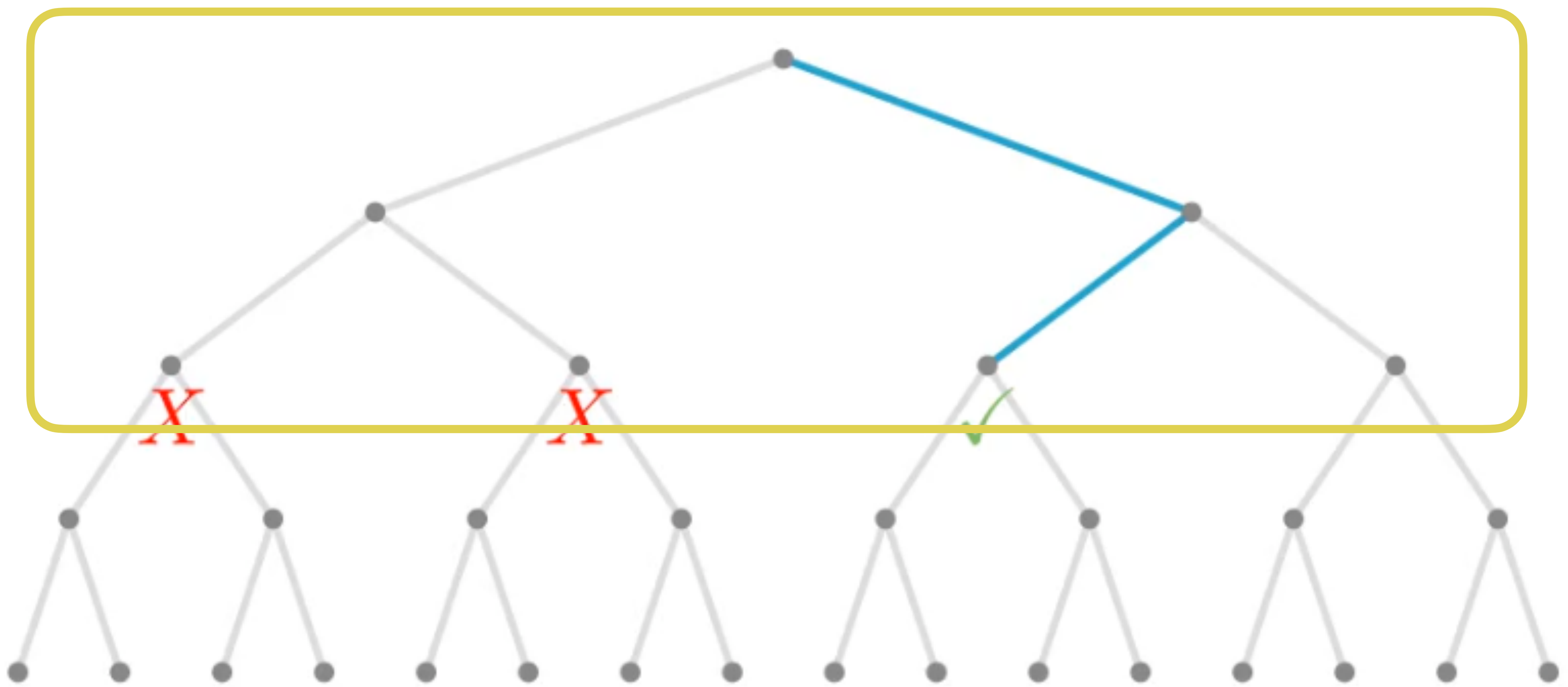
Simple algorithm

- Partial assignment
  - Gaussian elimination



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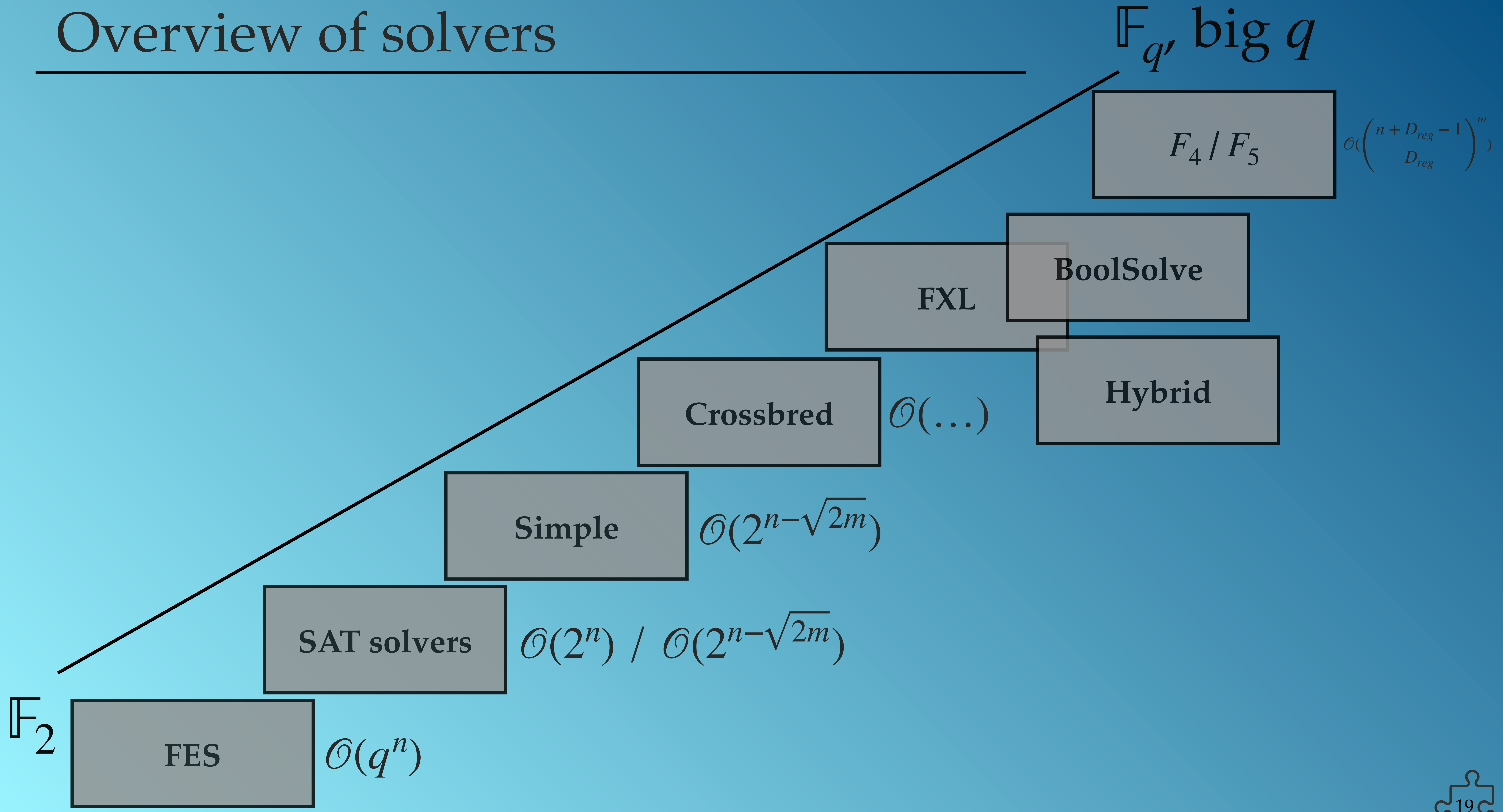
Simple algorithm

Guess sufficiently many variables so that the remaining polynomial system can be solved by linearization.

Overview of solvers



Gröbner basis algorithms

[Buchberger, 1965]

[Lazard, 1983]

F_4/F_5 [Faugère, 1999/2002]

(XL [Courtois, Klimov, Patarin, Shamir, 2000])

Gröbner basis algorithms (intuition)

*We are essentially describing the XL algorithm.

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	x_1x_2	x_1x_3	x_1x_4	x_1	x_2x_3	x_2x_4	x_2	x_3x_4	x_3	x_4	1
f_1	0	1	0	1	0	1	0	0	1	1	0
f_2	0	0	1	1	1	0	1	1	0	1	0
f_3	0	0	0	1	0	1	0	1	1	0	1
f_4	1	1	0	1	1	0	0	0	1	1	1
f_5	1	0	1	1	1	0	0	0	1	0	0
f_6	0	1	1	1	0	0	1	1	1	1	0

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	x_1x_2	x_1x_3	x_1x_4	x_1	x_2x_3	x_2x_4	x_2	x_3x_4	x_3	x_4	1
f_1	0	1	0	1	0	1	0	0	1	1	0
f_2	0	0	1	1	1	0	1	1	0	1	0
f_3	0	0	0	1	0	1	0	1	1	0	1
f_4	1	1	0	1	1	0	0	0	1	1	1
f_5	1	0	1	1	1	0	0	0	1	0	0
f_6	0	1	1	1	0	0	1	1	1	1	0

Gröbner basis algorithms (intuition)

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$D = 3$

$f_1 : x_1x_3 + x_2x_4 + x_1 + x_3 + x_4 = 0$
$f_2 : x_2x_3 + x_1x_4 + x_3x_4 + x_1 + x_2 + x_4 = 0$
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$f_5 : x_1x_2 + x_2x_3 + x_1x_4 + x_3 = 0$
$f_6 : x_1x_3 + x_1x_4 + x_3x_4 + x_1 + x_2 + x_3 + x_4 = 0$

	x_1x_2	x_1x_3	x_1x_4	x_1	x_2x_3	x_2x_4	x_2	x_3x_4	x_3	x_4	1	$x_1x_2x_3$	$x_1x_2x_4$	$x_1x_3x_4$	$x_2x_3x_4$
f_1	0	1	0	1	0	1	0	0	1	1	0				
f_2	0	0	1	1	1	0	1	1	0	1	0				
f_3	0	0	0	1	0	1	0	1	1	0	1				
f_4	1	1	0	1	1	0	0	0	1	1	1				
f_5	1	0	1	1	1	0	0	0	1	0	0				
f_6	0	1	1	1	0	0	1	1	1	1	0				
x_1f_1															
x_2f_1															
$\dots$															

Gröbner basis algorithms (intuition)

*We are essentially describing the XL algorithm.

$D = 4$

$f_1 : x_1x_3 + x_2x_4 + x_1 + x_3 + x_4 = 0$
$f_2 : x_2x_3 + x_1x_4 + x_3x_4 + x_1 + x_2 + x_4 = 0$
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	x_1x_2	x_1x_3	x_1x_4	x_1	x_2x_3	x_2x_4	x_2	x_3x_4	x_3	x_4	1	$x_1x_2x_3$	$x_1x_2x_4$	$x_1x_3x_4$	$x_2x_3x_4$	$x_1x_2x_3x_4$
f_1	0	1	0	1	0	1	0	0	1	1	0					
f_2	0	0	1	1	1	0	1	1	0	1	0					
f_3	0	0	0	1	0	1	0	1	1	0	1					
f_4	1	1	0	1	1	0	0	0	1	1	1					
f_5	1	0	1	1	1	0	0	0	1	0	0					
f_6	0	1	1	1	0	0	1	1	1	1	0					
x_1f_1																
x_2f_1																
…																
$x_1x_2f_1$																
$x_1x_3f_1$																

XL/Gröbner basis algorithms: complexity

XL / Gröbner basis algorithms: complexity

$$\mathcal{O}\left(mD_{reg}\binom{n+D_{reg}-1}{D_{reg}}^{\omega}\right)$$

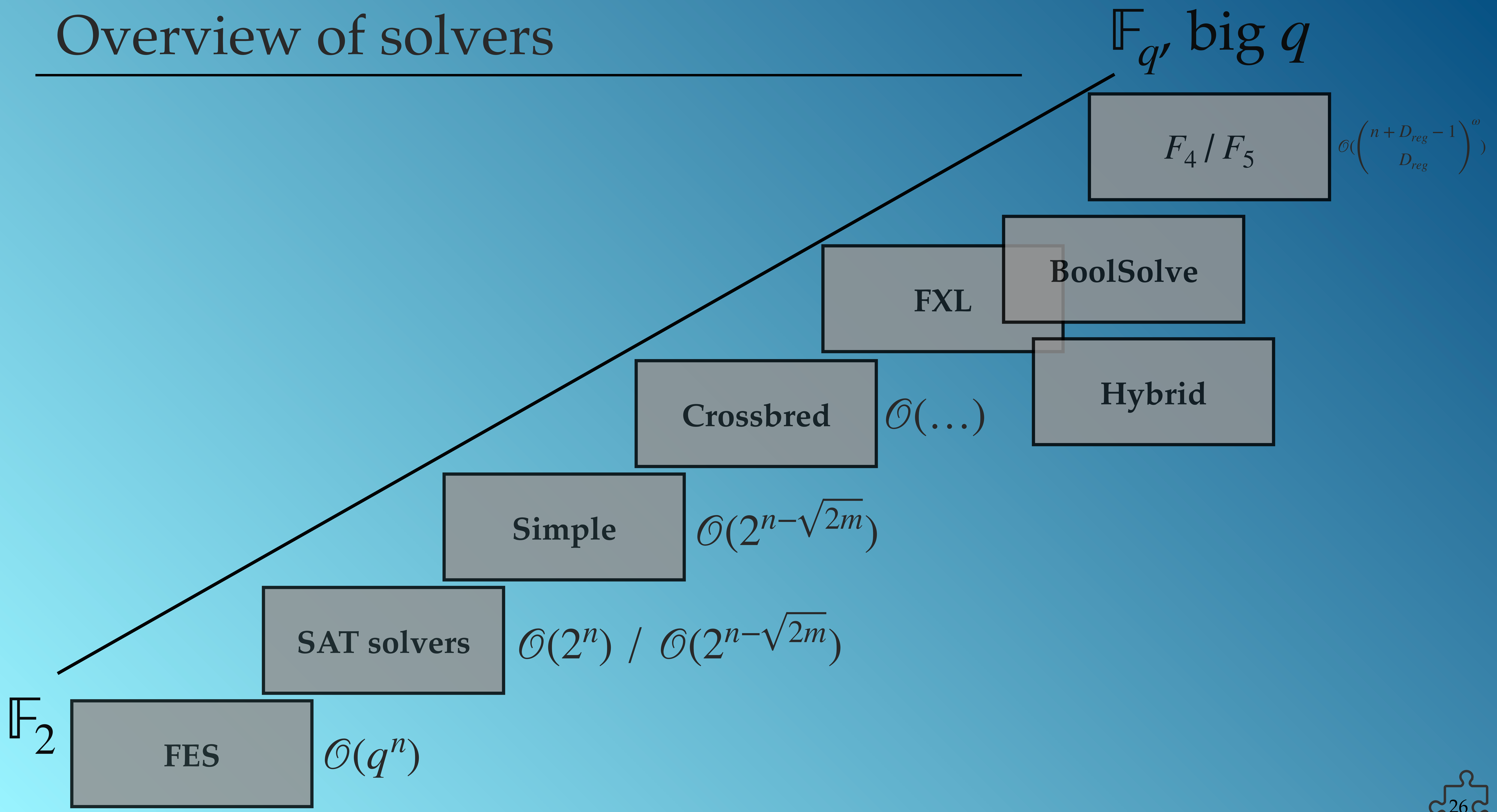
XL/Gröbner basis algorithms: complexity

$$\mathcal{O}\left(mD_{reg} \binom{n + D_{reg} - 1}{D_{reg}}^{\omega} \right)$$

D_{reg}: degree of regularity

↳ the power of the first non-positive coefficient in the expansion of $$\frac{(1-t^2)^m}{(1-t)^n}$$

Overview of solvers



Algebraic cryptanalysis: try it yourself !

Example.

Given matrices $\mathbf{C}_1, \mathbf{C}_2, \mathbf{D}_1, \mathbf{D}_2 \in \mathscr{M}_{n,n}(\mathbb{F}_q)$ (the space of matrices over $\mathbb{F}_q$ of size $n \times n$), find $\mathbf{A}, \mathbf{B} \in \mathrm{GL}_n(\mathbb{F}_q)$ (the space of invertible matrices over $\mathbb{F}_q$ of size $n \times n$), such that

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⟶ Demo

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→ Demo

→ In the assignment:

- Write down the equations;
- Find a better modelisation for this problem;

Modelisation

A motivating example: a better idea for modelisation.

Given matrices $\mathbf{C}_1, \mathbf{C}_2, \mathbf{D}_1, \mathbf{D}_2 \in \mathscr{M}_{n,n}(\mathbb{F}_q)$ (the space of matrices over $\mathbb{F}_q$ of size $n \times n$), find $\mathbf{A}, \mathbf{B} \in \mathrm{GL}_n(\mathbb{F}_q)$ (the space of invertible matrices over $\mathbb{F}_q$ of size $n \times n$), such that

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→ Results in a linear system with the same number of variables and equations.

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- Demo
- Results in a linear system with the same number of variables and equations.
  - If $\mathbf{C}_1, \mathbf{C}_2, \mathbf{D}_1, \mathbf{D}_2$ are all full rank, we should have a unique solution.
  - We can easily recover $\mathbf{A}$ from $\mathbf{A}^{-1}$.

Multivariate digital signature schemes

Multivariate signatures

Fiat-Shamir construction

trapdoor construction

Examples.

MQDSS

SOFIA

Examples.

HFEv-

UOV

The MQ problem (recall)

A quadratic system of m equations in n variables over a finite field $\mathbb{F}_q$:

$$f^{(k)}(x_1, \ldots, x_n) = \sum_{1 \le i \le j \le n} \gamma_{ij}^{(k)} x_i x_j + \sum_{1 \le i \le n} \beta_i^{(k)} x_i + \alpha^{(k)}$$

The MQ problem

Given m multivariate quadratic polynomials $f^{(1)}, \ldots, f^{(m)}$ of n variables over a finite field $\mathbb{F}_q$, find a tuple $\mathbf{x} = (x_1, \ldots, x_n)$ in $\mathbb{F}_q^n$, such that $f^{(1)}(\mathbf{x}) = \ldots = f^{(m)}(\mathbf{x}) = 0$.

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→ Hard in general (should be hard for randomly generated instances).

→ Can become easy if we have some structure (a trapdoor).

The trapdoor construction

The trapdoor construction

- Central map:
  $f : (x_1, \ldots, x_n) \in \mathbb{F}_q^n \to \left(f^{(1)}(x_1, \ldots, x_n), \ldots, f^{(m)}(x_1, \ldots, x_n)\right) \in \mathbb{F}_q^m$

- Two bijective linear (or affine) transformations:
  $\mathbf{S} \in \mathrm{GL}_n(\mathbb{F}_q)$ and $\mathbf{T} \in \mathrm{GL}_m(\mathbb{F}_q)$

- Public map:
  $p = \mathbf{T} \circ f \circ \mathbf{S}$

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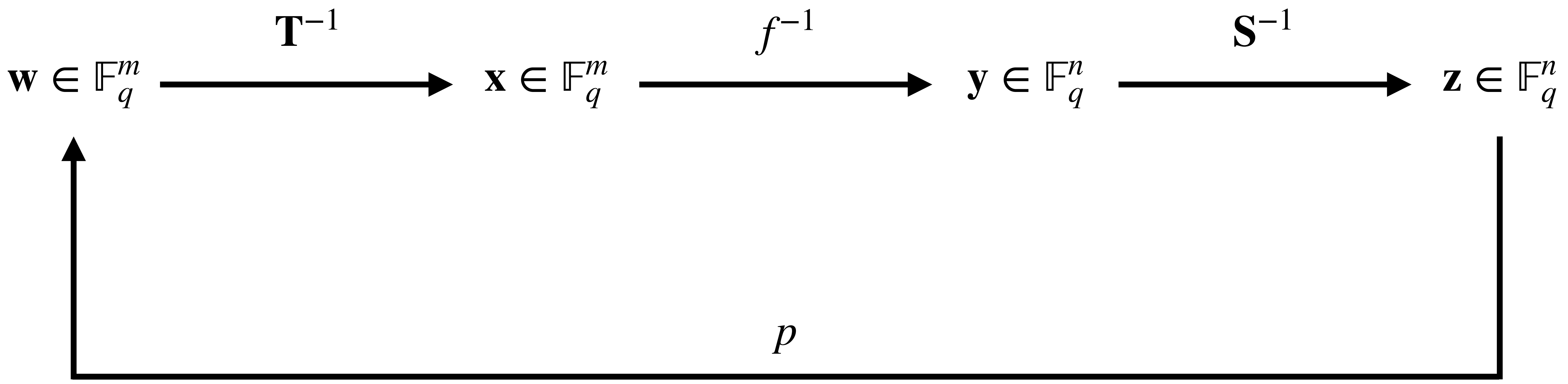
Main idea:

- The central map has a structure such that it is easy to find preimages: it is easy (polynomial time) to compute $f^{-1}(\mathbf{x})$ for a target vector $\mathbf{x}$.

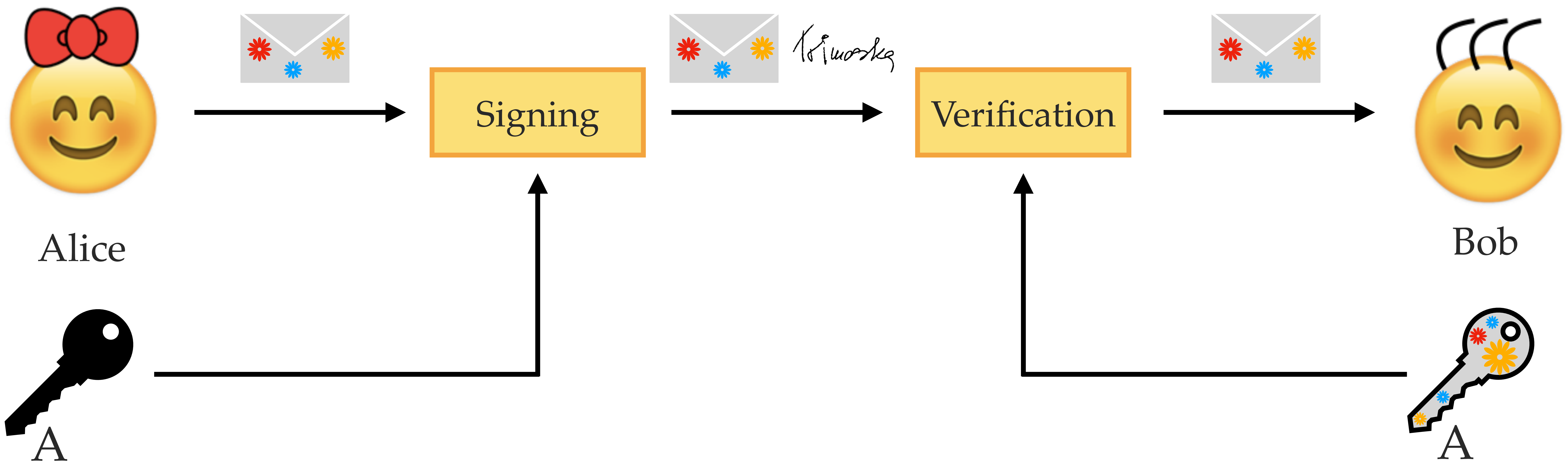
- The linear transformations hide the structure of the central map.

The trapdoor construction



General workflow

The trapdoor construction



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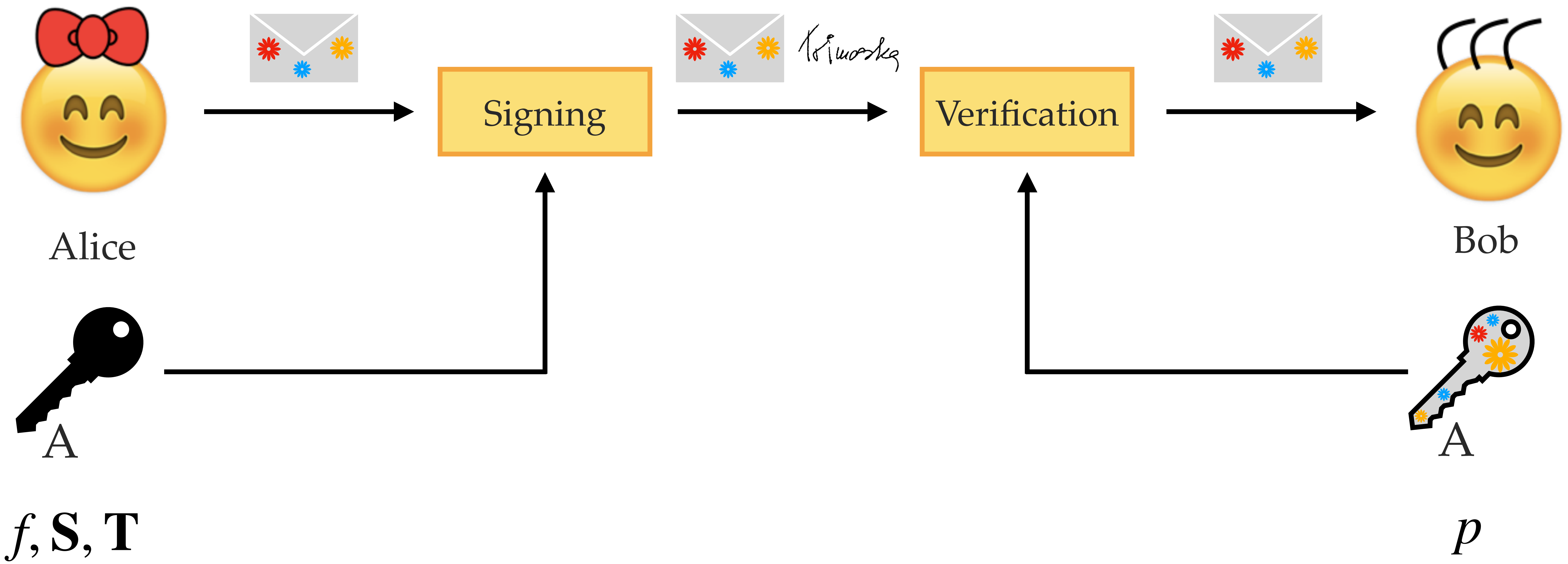

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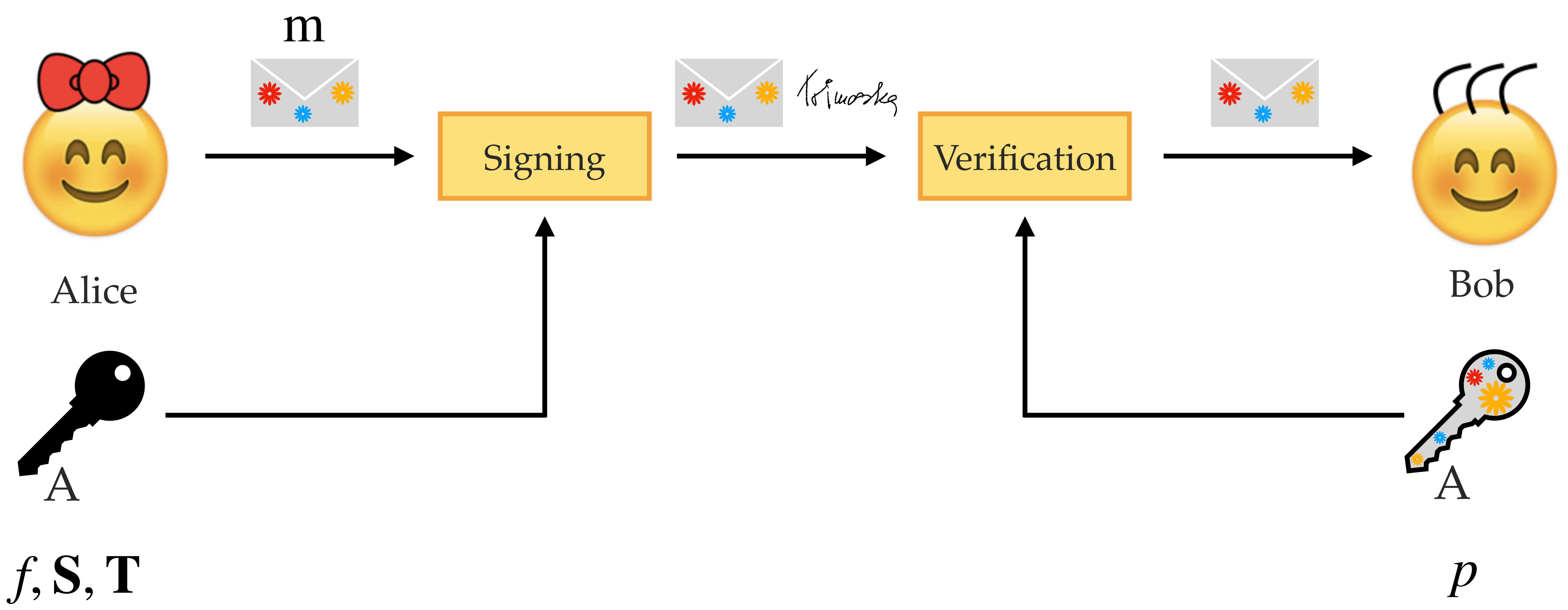

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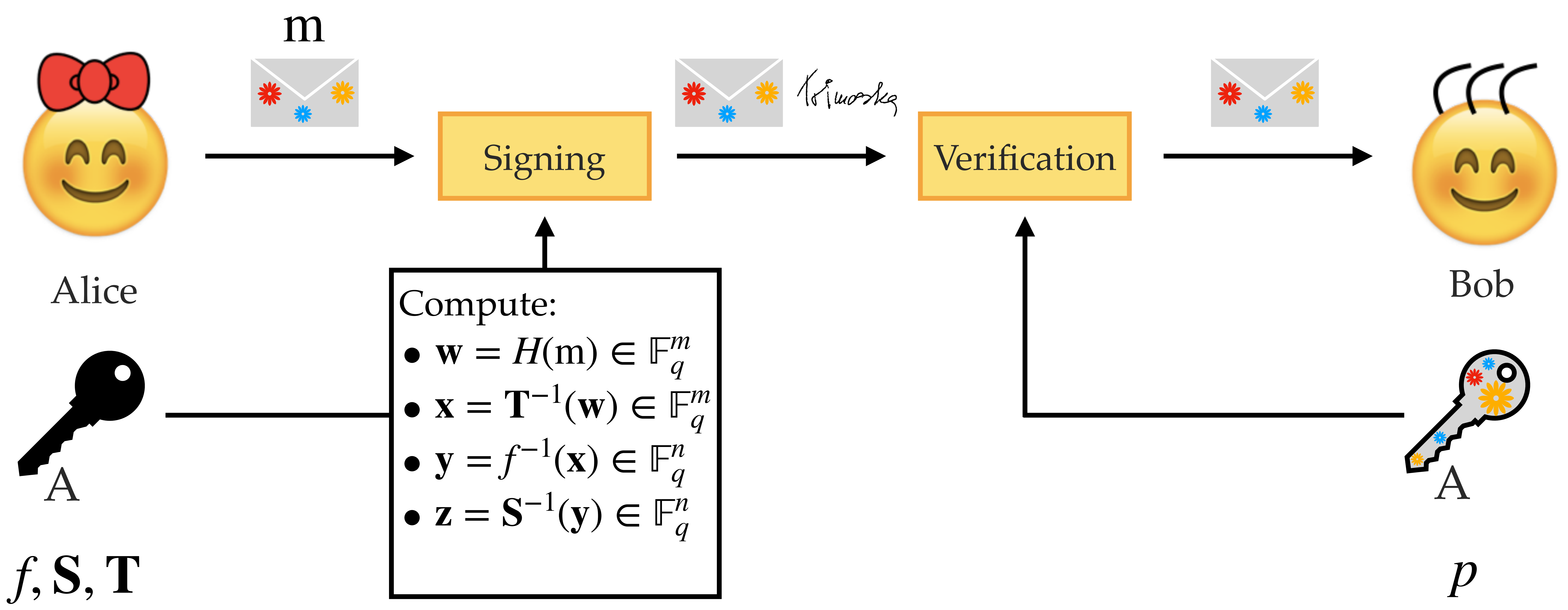

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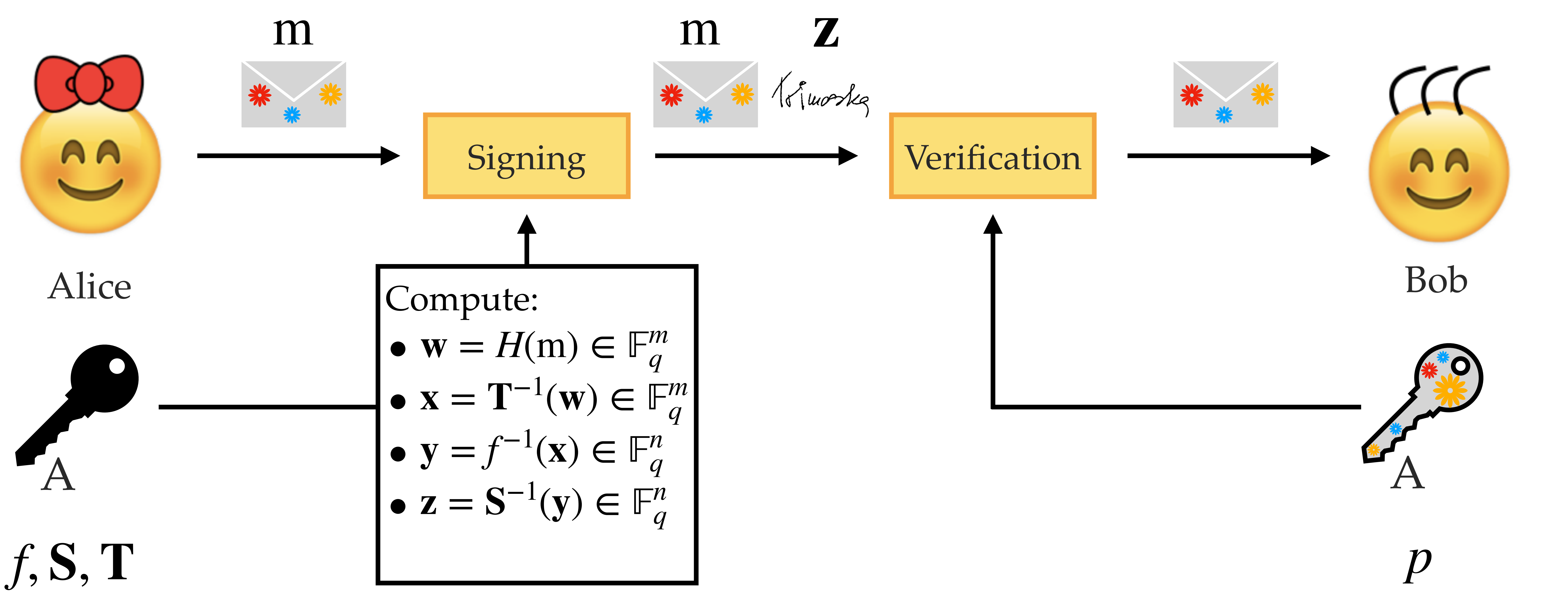

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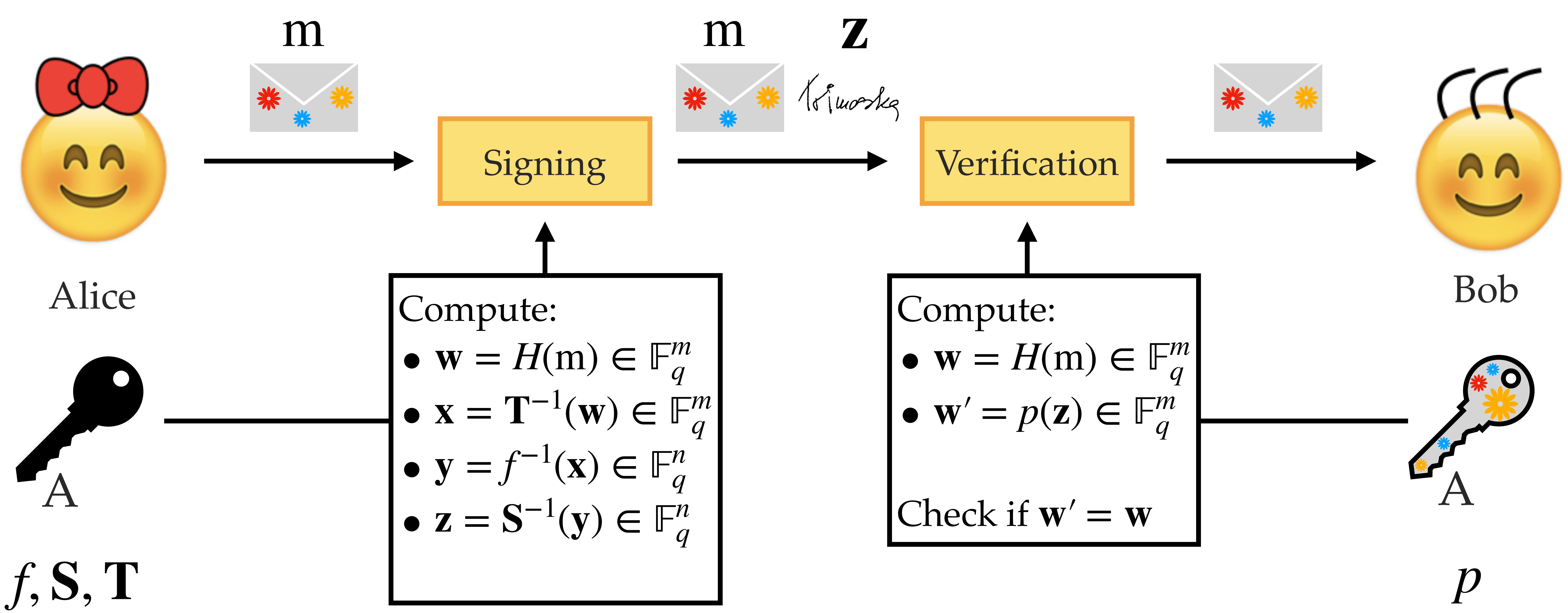

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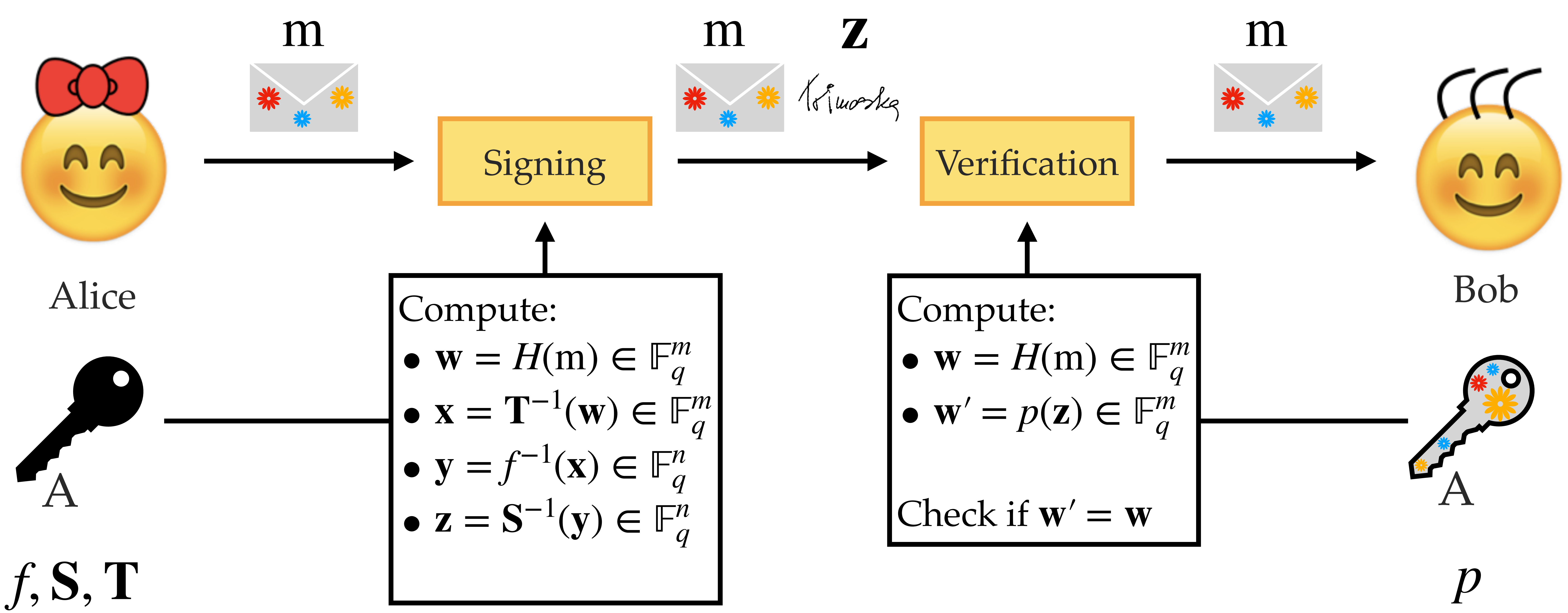

Unbalanced Oil and Vinegar (UOV)

[Kipnis, Patarin, Goubin, 1999]

The UOV central map

Unbalanced Oil and Vinegar [Kipnis, Patarin, Goubin, '99]

$$f^{(k)}(x_1, \ldots, x_n) = \sum_{i \in V, j \in V} \gamma_{ij}^{(k)} x_i x_j + \sum_{i \in V, j \in O} \gamma_{ij}^{(k)} x_i x_j + \sum_{i=1}^{n} \beta_i^{(k)} x_i + \alpha^{(k)}$$

Index set of vinegar variables: $V = \{1, \ldots, v\}$

Index set of oil variables: $O = \{v + 1, \ldots, n\}$

The UOV central map

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→ The central map is constructed in such a way that enumerating all of the vinegar variables leaves us with a linear system in the oil variables (oil does not mix with oil).

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- The central map is constructed in such a way that enumerating all of the vinegar variables leaves us with a linear system in the oil variables (oil does not mix with oil).
- Everything is as described in the previous slides, except that we do not have a linear transformation on the output: $\mathbf{T} = \mathbf{I}$.

Matrix representation of quadratic forms

Quadratic form: $f(\mathbf{x}) = \sum \gamma_{ij} x_i x_j$

$$\mathbf{x}^\top \quad \begin{bmatrix} x_1 & x_2 & x_3 & x_4 \end{bmatrix}$$

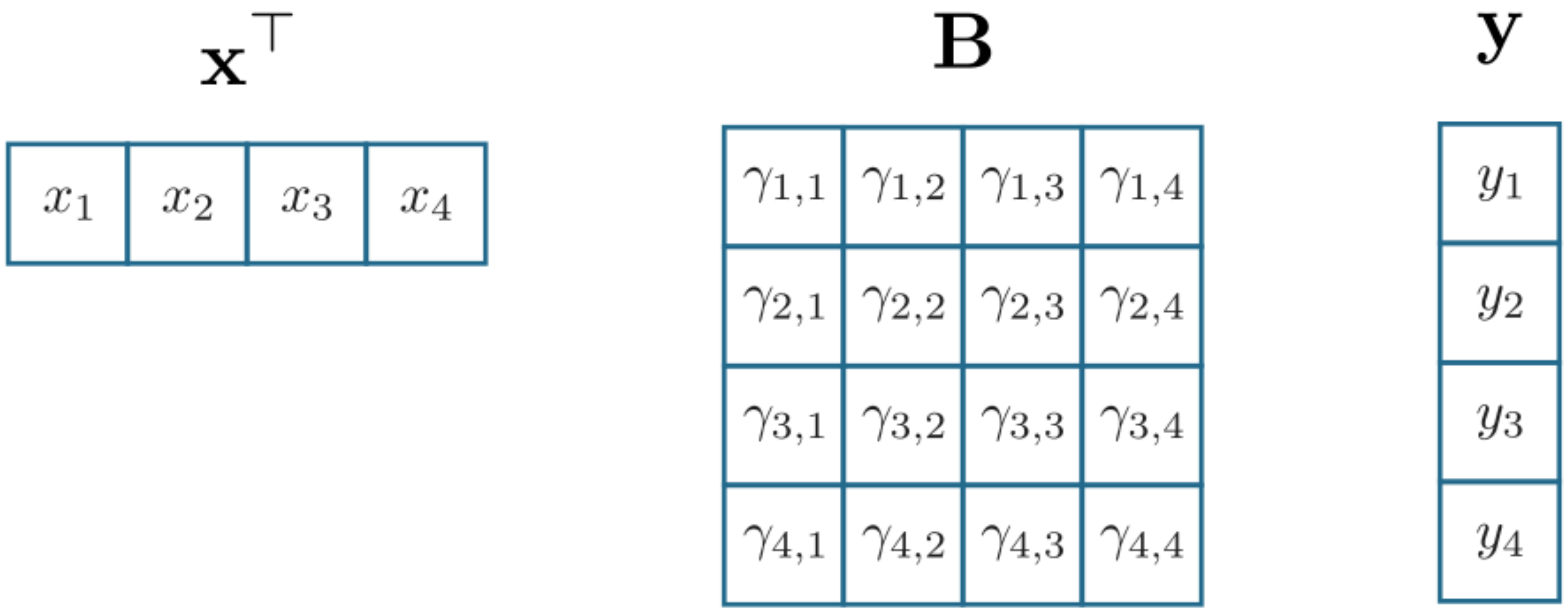
$$\mathbf{F} \quad \begin{bmatrix} \gamma_{1,1} & \frac{\gamma_{1,2}}{2} & \frac{\gamma_{1,3}}{2} & \frac{\gamma_{1,4}}{2} \\ \frac{\gamma_{2,1}}{2} & \gamma_{2,2} & \frac{\gamma_{2,3}}{2} & \frac{\gamma_{2,4}}{2} \\ \frac{\gamma_{3,1}}{2} & \frac{\gamma_{3,2}}{2} & \gamma_{3,3} & \frac{\gamma_{3,4}}{2} \\ \frac{\gamma_{4,1}}{2} & \frac{\gamma_{4,2}}{2} & \frac{\gamma_{4,3}}{2} & \gamma_{4,4} \end{bmatrix}$$

$$\mathbf{x} \quad \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$$

so with $\mathbf{x} = (x_1, \ldots, x_n)$, we get $\mathbf{x}^\top \mathbf{F} \mathbf{x}$.

Matrix representation of bilinear forms

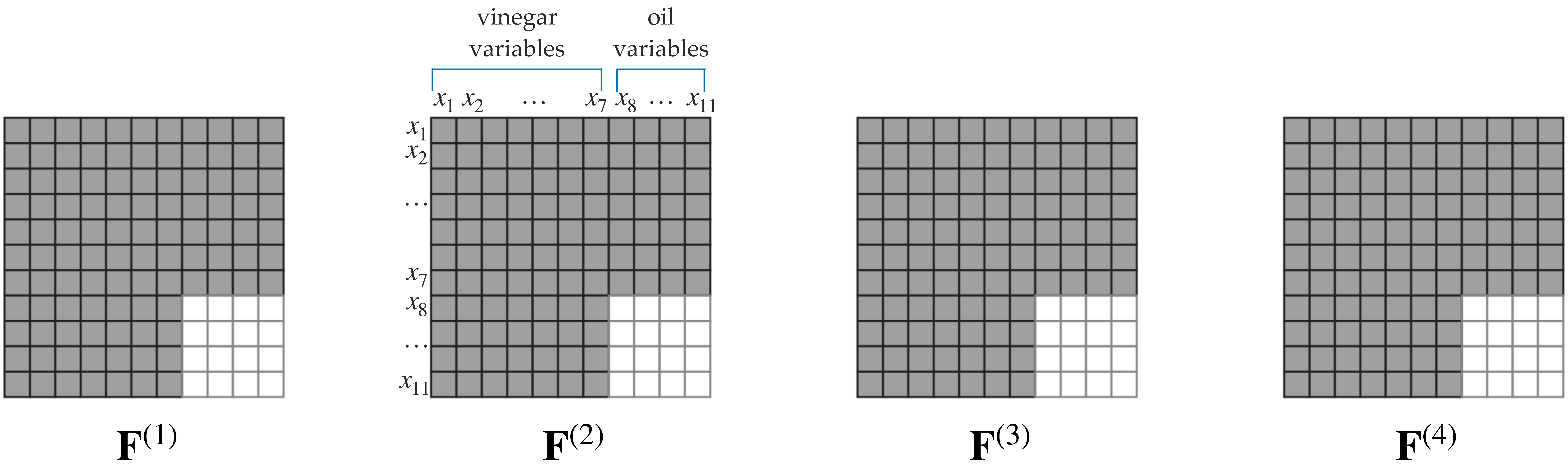
Bilinear form: $f(\mathbf{x}, \mathbf{y}) = \sum \gamma_{ij} x_i y_j$



so with $\mathbf{x} = (x_1, \ldots, x_n)$ and $\mathbf{y} = (y_1, \ldots, y_n)$, we get $\mathbf{x}^\top \mathbf{B} \mathbf{y}$.

The UOV central map

Toy example: $v = 7, m = 4$



*Grayed areas represent the entries that are possibly nonzero; blank areas denote the zero entries;

UOV key generation

In matrix representation

$$\mathbf{P}^{(k)} = \mathbf{S}^{\top}\mathbf{F}^{(k)}\mathbf{S}, \text{ for all } k \in \{1, \ldots, m\}.$$

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UOV in the NIST competition

UOV
TUOV
PROV
MAYO
VOX
QR-UOV
SNOVA

UOV in the NIST competition

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- MQ-Sign (in KpqC)

Example.

	NIST SL	n	m	$\mathbb{F}_q$	|pk| (bytes)	|sk| (bytes)	|cpk| (bytes)	|sig+salt| (bytes)
ov-Ip	1	112	44	$\mathbb{F}_{256}$	278 432	237 896	43 576	128
ov-Is	1	160	64	$\mathbb{F}_{16}$	412 160	348 704	66 576	96
ov-III	3	184	72	$\mathbb{F}_{256}$	1 225 440	1 044 320	189 232	200
ov-V	5	244	96	$\mathbb{F}_{256}$	2 869 440	2 436 704	446 992	260

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- We choose $n \sim 2.5m$ (slightly bigger than)

UOV-like schemes have:

- Big public keys
- Small signatures

Attacks on UOV

Attacks on UOV

- Direct attack
- Reconciliation attack
- Kipnis-Shamir attack
- Intersection attack

Direct attack

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Try to forge a signature with only the knowledge of the public key.

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Reconciliation attack

[Ding, Yang, Chen, Chen, Cheng, 2008]
(using description from [Samardjiska, Gligoroski, 2014])

The secret subspace O

The map p with a UOV trapdoor vanishes on a linear subspace $O \subset \mathbb{F}_q^n$ of $\dim(O) = m$:

$$p(\mathbf{o}) = 0, \text{ for all } \mathbf{o} \in O.$$

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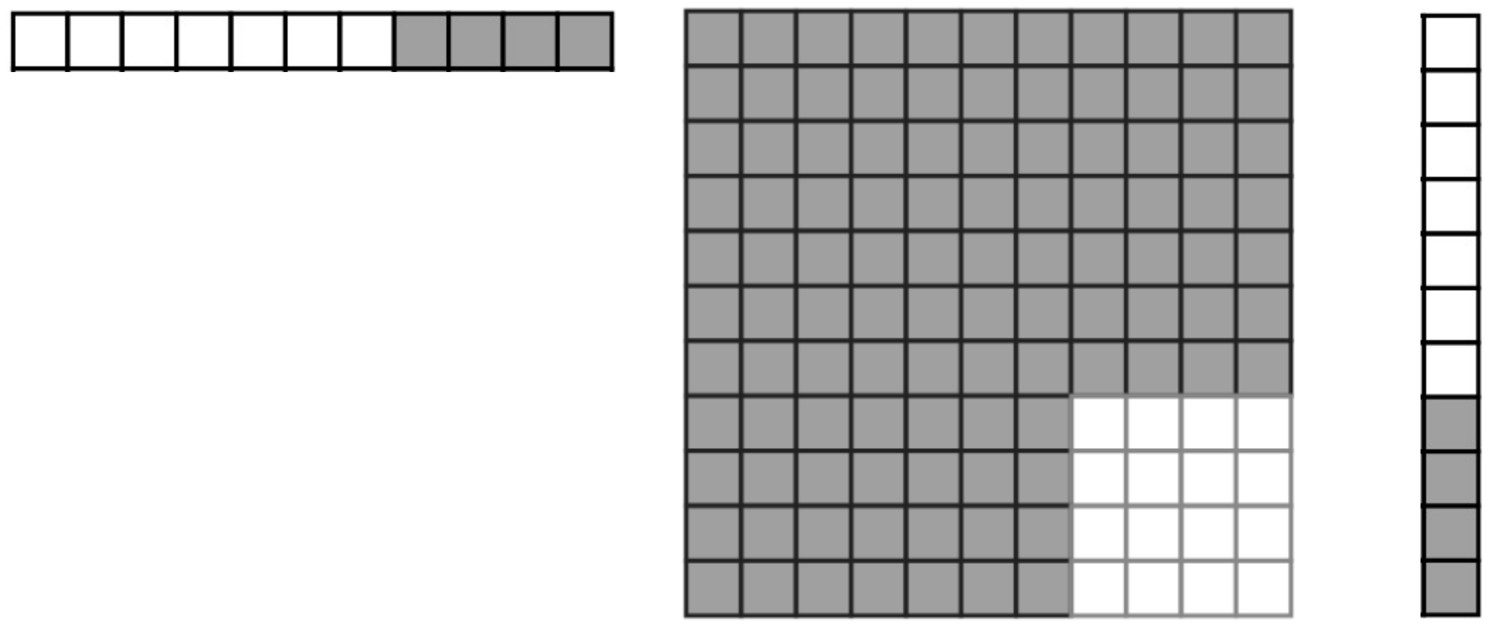
Let $O' \in \mathbb{F}_q^n$ be the m-dimensional space that consists of all the vectors whose first $n - m$ entries (corresponding to the vinegar variables) are zero: $O' = \{\mathbf{v} \mid v_i = 0 \text{ for all } i \leq n - m\}$.



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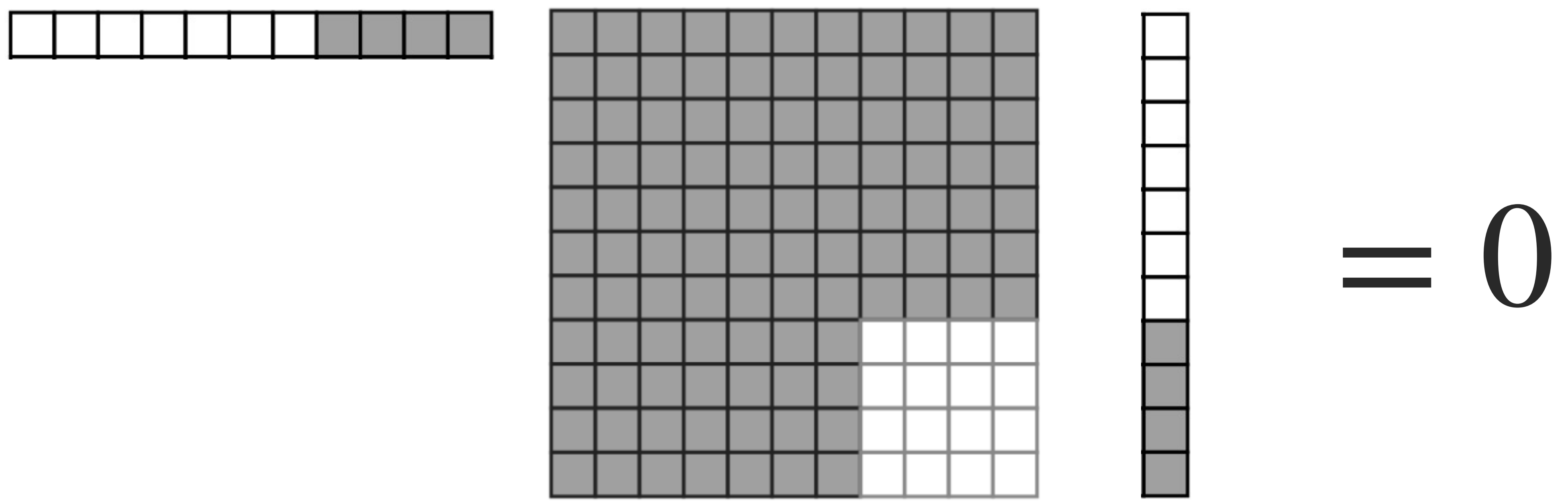

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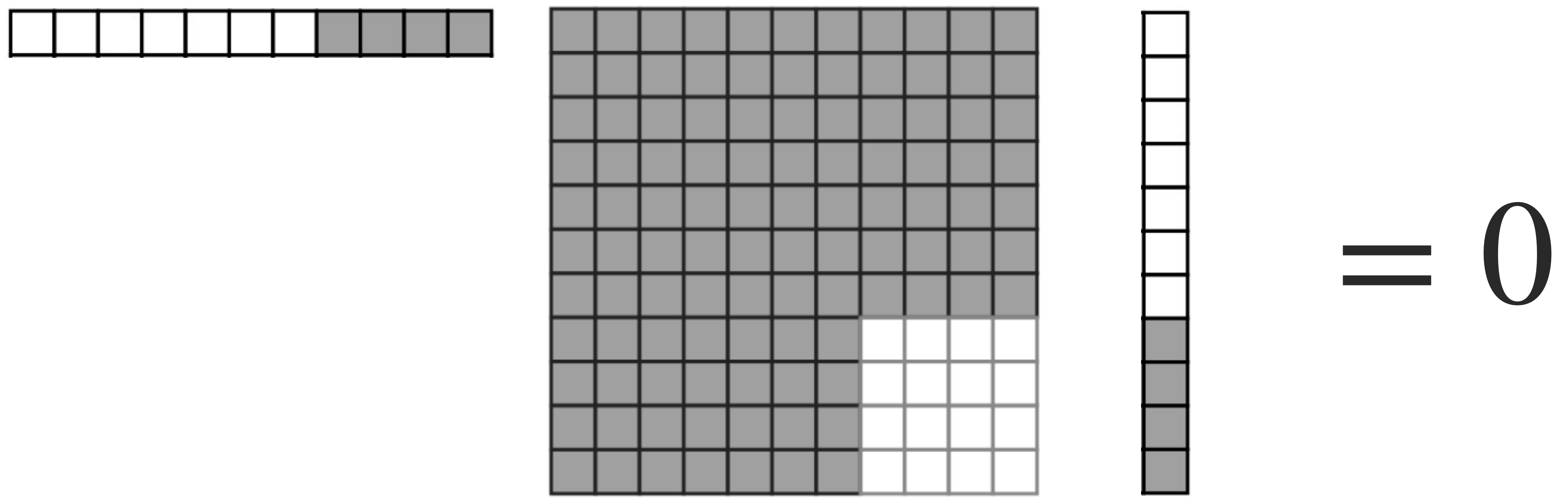

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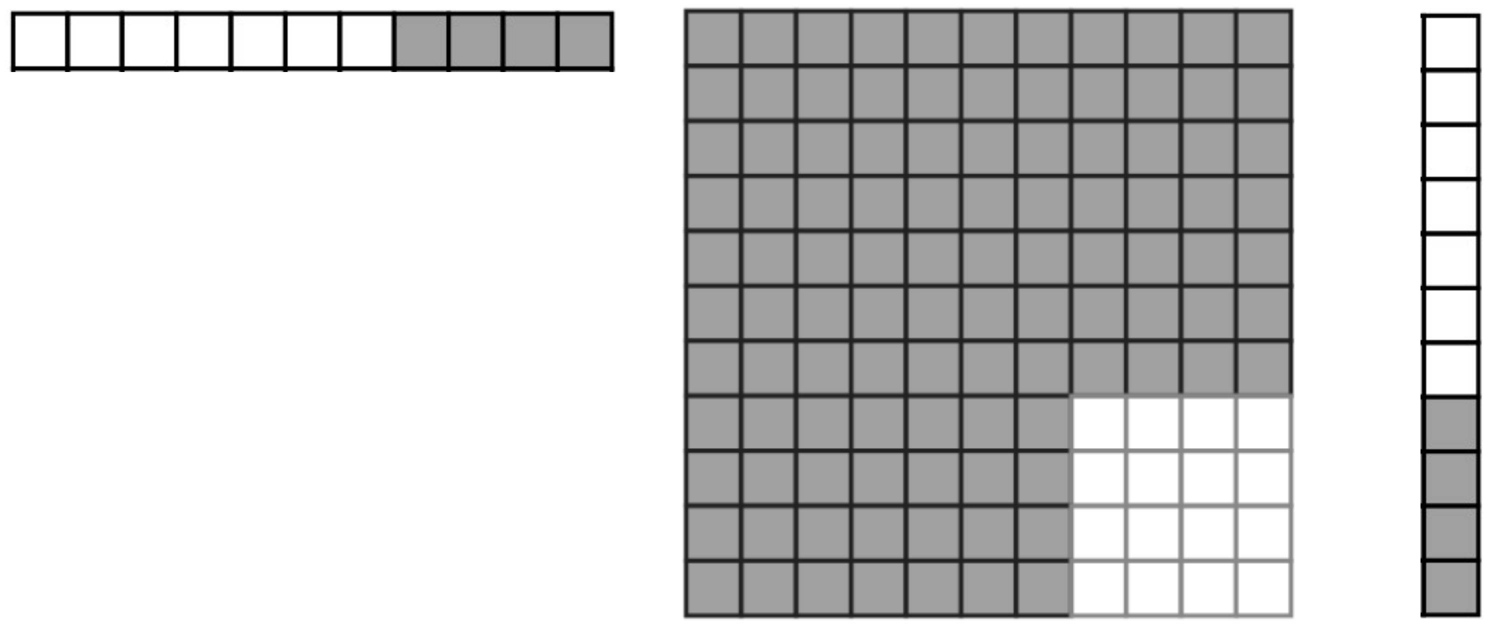

$= 0$

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p vanishes on O.

Reconciliation attack

Find the secret oil subspace O : find m linearly independent vectors in O.

The polar form

The polar form of a quadratic map $p = (p^{(1)}, \ldots, p^{(m)})$ is the bilinear form $p' = (p^{'(1)}, \ldots, p^{'(m)})$ such that

$$p^{'(k)}(\mathbf{x}, \mathbf{y}) = p^{(k)}(\mathbf{x} + \mathbf{y}) - p^{(k)}(\mathbf{x}) - p^{(k)}(\mathbf{y}), \text{ for all } k \in \{1, \ldots, m\} .$$

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What does $p'^{(k)}(\mathbf{x}, \mathbf{y})$ look like ?

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→ So, p' is bilinear and symmetric.

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Find the secret oil subspace O : find m linearly independent vectors in O.

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→ Equations:

For $i \in \{1,\ldots,m\}$ **do**

$\mathbf{o}_i = (o_1, \ldots, o_v, 0, \ldots, 1_{n-i+1}, 0, \ldots, 0)$

Solve:

$$\mathbf{o}_i^\top \mathbf{B}^{(k)} \mathbf{o}_j = 0, \text{ for } k \in \{1,\ldots,m\} \text{ and } j < i$$

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Kipnis-Shamir attack

[Kipnis, Shamir, 1998]

The orthogonal complement of a subspace

Let $V \subset \mathbb{F}_q^n$. The orthogonal complement of V is $V^\perp$ such that

$$V^\perp = \{\tilde{\mathbf{v}}_i \in \mathbb{F}_q^n \,|\, \langle \mathbf{v}_j, \tilde{\mathbf{v}}_i \rangle = 0, \text{ for all } \mathbf{v}_j \in V\}.$$

If V is m-dimensional, then $V^\perp$ is $(n-m)$-dimensional.

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→ Finding a common invariant subspace of a large number of linear maps is easy.

→ Oil and Vinegar becomes Unbalanced Oil and Vinegar because of this attack.

Intersection attack

[Beullens, 2021]

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Find the secret oil subspace O. Use the ideas of the Kipnis-Shamir attack, but for the unbalanced case ($n > 2m$).

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→ The attack can be generalised to find a vector in the intersection of more than two subspaces.

Recap

- The MQ problem is (usually) hard.
- We have a variety of solvers for (over)determined systems.
- Modelisation can be crucial to how efficient an attack is.
- The MQ problem can be easy for some structured systems. We use this to build trapdoors in crypto.
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Resources at https://mtrimoska.com/QSI-multivariate/

Credits: created with ♥, and Manim.